

TOPOLOGICAL K -THEORY OF THE GROUP C^* -ALGEBRA OF A SEMI-DIRECT PRODUCT $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ FOR A FREE CONJUGATION ACTION

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ABSTRACT. We compute the topological K -theory of the group C^* -algebra $C_r^*(\Gamma)$ for a group extension $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow \mathbb{Z}/m \rightarrow 1$ provided that the conjugation action of \mathbb{Z}/m on \mathbb{Z}^n is free outside the origin.

INTRODUCTION

Throughout this paper let $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow \mathbb{Z}/m \rightarrow 1$ be a group extension such that conjugation action of \mathbb{Z}/m on \mathbb{Z}^n is free outside the origin. Our main goal is to compute the topological K -theory of the group C^* -algebra $C_r^*(\Gamma)$. This generalizes results of Davis-Lück [7], where m was assumed to be a prime. Except ideas from that paper, the proof of a Conjecture due to Adem-Ge-Pan-Petrosyan in Langer-Lück [11] is a key ingredient. The calculation and its result are surprisingly complicated. It will play an important role in a forthcoming paper by Li-Lück [12]. There the computation of the topological K -theory of a C^* -algebra associated to the ring of integers in an algebraic number field will be carried out in general, thus generalizing the work of Cuntz and Li [6] who had to assume that $+1$ and -1 are the only roots of unity.

0.1. Main Result. Our main result is the following theorem. In the sequel $C_r^*(G)$ is the *reduced group C^* -algebra* of a group G . We denote by \underline{EG} the *classifying space of proper actions* of a group G and by \underline{BG} its quotient space $G \backslash \underline{EG}$. Let $\hat{H}^*(G; M)$ be the *Tate cohomology* of a group G with coefficients in a $\mathbb{Z}G$ -module M . Denote by $\Lambda^i \mathbb{Z}^n$ the i -th exterior power.

Theorem 0.1 (Computation of the topological K -theory). *Consider the extension of groups $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow \mathbb{Z}/m \rightarrow 1$ such that the conjugation action of \mathbb{Z}/m on \mathbb{Z}^n is free outside the origin $0 \in \mathbb{Z}^n$. Let \mathcal{M} be the set of conjugacy classes of maximal finite subgroups of Γ .*

(i) *We obtain an isomorphism*

$$\omega_1: K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(\underline{B}\Gamma).$$

Restriction with the inclusion $k: \mathbb{Z}^n \rightarrow \Gamma$ induces an isomorphism

$$k^*: K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}.$$

Induction with the inclusion k yields a homomorphism

$$\overline{k_*}: \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \rightarrow K_1(C_r^*(\Gamma)).$$

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It fits into an exact sequence

$$0 \rightarrow \widehat{H}^{-1}(\mathbb{Z}/m, K_1(C_r^*(\mathbb{Z}^n))) \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \xrightarrow{\overline{k}_*} K_1(C_r^*(\Gamma)) \rightarrow 0.$$

In particular \overline{k}_* is surjective and its kernel is annihilated by multiplication with m ;

(ii) There is an exact sequence

$$0 \rightarrow \bigoplus_{(M) \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \xrightarrow{\bigoplus_{(M) \in \mathcal{M}} i_M} K_0(C_r^*(\Gamma)) \xrightarrow{\omega_0} K_0(\underline{B}\Gamma) \rightarrow 0,$$

where $\widetilde{R}_{\mathbb{C}}(M)$ is the kernel of the map $R_{\mathbb{C}}(M) \rightarrow \mathbb{Z}$ sending the class $[V]$ of a complex M -representation V to $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}M} V)$ and the map i_M comes from the inclusion $M \rightarrow \Gamma$ and the identification $R_{\mathbb{C}}(M) = K_0(C_r^*(M))$.

We obtain a homomorphism

$$\overline{k}_* \oplus \bigoplus_{(M) \in \mathcal{M}} i_M : \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_0(C_r^*(\mathbb{Z}^n)) \oplus \bigoplus_{(M) \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \rightarrow K_0(C_r^*(\Gamma)).$$

It is injective. It is bijective after inverting m ;

(iii) We have

$$K_i(C_r^*(\Gamma)) \cong \mathbb{Z}^{s_i}$$

where

$$s_i = \begin{cases} (\sum_{(M) \in \mathcal{M}} (|M| - 1)) + \sum_{l \in \mathbb{Z}} \text{rk}_{\mathbb{Z}}((\Lambda^{2l} \mathbb{Z}^n)^{\mathbb{Z}/m}) & \text{if } i \text{ even;} \\ \sum_{l \in \mathbb{Z}} \text{rk}_{\mathbb{Z}}((\Lambda^{2l+1} \mathbb{Z}^n)^{\mathbb{Z}/m}) & \text{if } i \text{ odd;} \end{cases}$$

(iv) If m is even, then $s_1 = 0$ and

$$K_1(C_r^*(\Gamma)) \cong \{0\}.$$

Another interesting result is Theorem 3.3 where we will address the cohomology of Γ and of the associated toroidal quotient $\underline{B}\Gamma = \Gamma \backslash \mathbb{R}^n$.

0.2. Organization of the paper. We will compute $K_*(C_r^*(G))$ in a more general setting in Section 1, where we consider groups G for which each non-trivial finite subgroup is contained in a unique maximal finite subgroup. In Section 2 we consider the special case where we additionally assume that the normalizer of every maximal finite subgroup is the maximal finite subgroup itself. The groups Γ appearing in Theorem 0.1 will satisfy this assumption.

In Sections 3 and 4 we deal with the cohomology and the topological K -theory of the spaces $B\Gamma$ and $\underline{B}\Gamma$, and, finally complete the proof of Theorem 0.1.

The rest of the paper is devoted to the computation of the numbers s_i appearing in Theorem 0.1. Recall that \mathbb{Z}^n becomes a $\mathbb{Z}[\mathbb{Z}/m]$ -module by the conjugation action. Sometimes we write \mathbb{Z}_ρ^n instead of \mathbb{Z}^n to emphasize the $\mathbb{Z}[\mathbb{Z}/m]$ -module structure. Notice that the numbers s_i are determined by the set \mathcal{M} of conjugacy classes of maximal finite subgroups of Γ and the two numbers $\sum_{l \geq 0} \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}_\rho^n)^{\mathbb{Z}/m})$ and $\sum_{l \geq 0} (-1)^l \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}_\rho^n)^{\mathbb{Z}/m})$.

Section 5 is devoted to compute the partial ordered set of conjugacy classes of finite subgroups of Γ , directed by subconjugation, in terms of group cohomology. This yields also a computation of the set \mathcal{M} .

In Section 6 we will compute \mathcal{M} and the numbers $\sum_{l \geq 0} \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}_\rho^n)^{\mathbb{Z}/m})$ in the case, where m is a prime power.

The general case is treated in Section 7. The calculation of \mathcal{M} is explicit. The numbers $\sum_{l \geq 0} \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}_\rho^n)^{\mathbb{Z}/m})$ will be computed explicitly provided that m is even. For m odd our methods yield at least a recipe for a case by case computation, the problem is to determine $\sum_{l \geq 0} \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}_\rho^n)^{\mathbb{Z}/m})$. Since the roots of unity in an

algebraic number field is always a finite cyclic group of even order, the case, where m is even, is the most interesting for us.

In Section 8 we compute the equivariant \mathbb{Z}/m -Euler characteristic of $\mathbb{Z}^n \backslash \underline{E}\Gamma$ which takes values in the Burnside ring of \mathbb{Z}/m . This will determine explicitly the numbers $\sum_{l \geq 0} (-1)^l \cdot \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}_\rho^n)^{\mathbb{Z}/m})$.

The actual answers to the computations of the set \mathcal{M} and the numbers s_i are interesting but also very complicated. As an illustration we state and explain some examples already here in the introduction. The group Γ and the numbers m and n are the ones appearing in Theorem 0.1. Here and in the sequel we will use the convention that a sum of real numbers indexed by the empty set is understood to be zero, e.g., $\sum_{i=2}^1 a_i$ is defined to be zero.

0.3. m is a prime. Suppose that $m = p$ for a prime number p . We have $\mathbb{Z}_\rho^n \otimes_{\mathbb{Q}} \mathbb{Q} \cong_{\mathbb{Q}[\mathbb{Z}/p]} \mathbb{Q}[\zeta_p]^k$ for $\zeta_p = \exp(2\pi i/p)$ and some natural number k . The natural number k is determined by the property $n = (p-1) \cdot k$. All non-trivial finite subgroups of Γ are cyclic of order p and are maximal finite. We obtain from Theorem 6.18, Lemma 6.28 and Theorem 8.7 (ii)

$$\begin{aligned} |\mathcal{M}| &= p^k; \\ \sum_{(M) \in \mathcal{M}} (|M| - 1) &= p^k \cdot (p-1); \\ \sum_{l \in \mathbb{Z}} \text{rk}_{\mathbb{Z}}((\Lambda^l L)^{\mathbb{Z}/m}) &= \begin{cases} 1 + \frac{2^n - 1}{p} & p \neq 2; \\ 2^{k-1} & p = 2; \end{cases} \\ \sum_{l \in \mathbb{Z}} (-1)^l \cdot \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}^n)^{\mathbb{Z}/m}) &= p^{k-1} \cdot (p-1). \end{aligned}$$

This implies (and is consistent with [7])

$$s_i = \begin{cases} p^k \cdot (p-1) + \frac{2^n + p - 1}{2p} + \frac{p^{k-1} \cdot (p-1)}{2} & p \neq 2 \text{ and } i \text{ even}; \\ \frac{2^n + p - 1}{2p} - \frac{p^{k-1} \cdot (p-1)}{2} & p \neq 2 \text{ and } i \text{ odd}; \\ 3 \cdot 2^{k-1} & p = 2 \text{ and } i \text{ even}; \\ 0 & p = 2 \text{ and } i \text{ odd}. \end{cases}$$

0.4. m is a prime power p^r for $r \geq 2$. Next we consider the case, where m is a prime power, let us say $m = p^r$. Since we have treated the case $r = 1$ already in Example 0.3, we will assume in the sequel $r \geq 2$. There exists precisely one natural number k satisfying $n = (p-1) \cdot p^{r-1} \cdot k$. We obtain from Theorem 6.18 and Remark 8.9

$$\begin{aligned} \sum_{(M) \in \mathcal{M}} (|M| - 1) &= \sum_{j=1}^r \sum_{(M) \in \mathcal{M}(G_j)} (p^j - 1) \\ &= p^k \cdot (p^r - 1) + \sum_{j=1}^{r-1} (p^{kp^{r-j} - r + j} - p^{kp^{r-j-1} - r + j}) \cdot (p^j - 1). \end{aligned}$$

We get from Lemma 6.28

$$\sum_{l \geq 0} \text{rk}_{\mathbb{Z}}(\Lambda^l L)^{\mathbb{Z}/m} = 1 + \frac{2^n - 1}{p^r} \quad \text{if } p \neq 2.$$

If $p = 2$, Lemma 6.28 yields

$$\sum_{l \geq 0} \text{rk}_{\mathbb{Z}}(\Lambda^l L)^{\mathbb{Z}/m} = \begin{cases} 2^{2k-2} + 2^{k-1} & \text{if } r = 2; \\ 2^{k-1} + 2^{k \cdot 2^{r-2} - r + 1} + 2^{k \cdot 2^{r-1} - r} + \sum_{i=3}^{r-1} 2^{k \cdot 2^{r-i} - r + i - 1} & \text{if } r \geq 3. \end{cases}$$

We get from Theorem 6.18, Theorem 8.7 (ii) and Remark 8.9

$$\begin{aligned} \sum_{l \geq 0} (-1)^l \cdot \text{rk}_{\mathbb{Z}}(\Lambda^l L)^{\mathbb{Z}/m} &= p^k \cdot \frac{(p^r - 1)}{p^r} + \sum_{j=1}^{r-1} (p^{kp^{r-j}-r+j} - p^{kp^{r-j-1}-r+j}) \cdot \frac{p^j - 1}{p^j} \\ &= p^k - p^{k-r} + \sum_{j=1}^{r-1} (p^{kp^{r-j}-r} - p^{kp^{r-j-1}-r}) \cdot (p^j - 1). \end{aligned}$$

We leave it to the reader to combine these results to determine the numbers s_i .

Example 0.2 ($m = 4$). We get for $m = 4$, i.e., $p = 2$ and $r = 2$

$$s_i = \begin{cases} 3 \cdot 2^{2k-2} + 3 \cdot 2^k & i \text{ even;} \\ 0 & i \text{ odd.} \end{cases}$$

Example 0.3 ($m = 9$). We get for $m = 9$, i.e., $p = 3$ and $r = 2$

$$s_i = \begin{cases} \frac{64^k + 8}{18} + 23 \cdot 3^{k-1} + 7 \cdot 3^{3k-2} & i \text{ even;} \\ \frac{64^k + 8}{18} - 3^{k-1} - 3^{3k-2} & i \text{ odd.} \end{cases}$$

0.5. m is square-free and even. Next we consider the case, where m is square-free and even. The case $m = 2$ has already been treated in Example 0.3. Hence we will assume in the sequel that $m = p_1 \cdot p_2 \cdots p_s$ for pairwise distinct prime numbers p_1, p_2, \dots, p_s with $p_1 = 2$ and $s \geq 2$. Then we get from Example 7.7 and Remark 8.9

$$\sum_{(M) \in \mathcal{M}} (|M| - 1) = m - 1 + \sum_{j=1}^s \frac{p_j \cdot (p_j - 1) \cdot (p_j^{n/(p_j-1)} - 1)}{m}.$$

From Lemma 7.9 we get $(\Lambda^{2l+1} L)^{\mathbb{Z}/m} = \{0\}$ for every $l \geq 0$. We conclude from Example 7.7, Theorem 8.7 (ii) and Remark 8.9

$$\sum_{l \in \mathbb{Z}} (-1)^l \cdot \text{rk}_{\mathbb{Z}}((\Lambda^l L)^{\mathbb{Z}/m}) = \frac{m-1}{m} + \sum_{j=1}^s \frac{(p_j - 1) \cdot (p_j^{n/(p_j-1)} - 1)}{m}.$$

Hence we get for i even

$$\begin{aligned} s_i &= m - 1 + \sum_{j=1}^s \frac{p_j \cdot (p_j - 1) \cdot (p_j^{n/(p_j-1)} - 1)}{m} + \frac{m-1}{m} + \sum_{j=1}^s \frac{(p_j - 1) \cdot (p_j^{n/(p_j-1)} - 1)}{m} \\ &= m + \frac{-m + (m-1) + \sum_{j=1}^s (p_j \cdot (p_j - 1) \cdot (p_j^{n/(p_j-1)} - 1) + (p_j - 1) \cdot (p_j^{n/(p_j-1)} - 1))}{m} \\ &= m + \frac{-1 + \sum_{j=1}^s (p_j^2 - 1) \cdot (p_j^{n/(p_j-1)} - 1)}{m}. \end{aligned}$$

Thus we have computed

$$s_i = \begin{cases} m + \frac{-1 + \sum_{j=1}^s (p_j^2 - 1) \cdot (p_j^{n/(p_j-1)} - 1)}{m} & i \text{ even;} \\ 0 & i \text{ odd.} \end{cases}$$

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1. GROUPS SATISFYING CONDITION $(M_{\mathcal{T}_R \subseteq \mathcal{F}_{in}})$

Let G be a group. A group $H \subseteq G$ is called *maximal finite* if H is finite and for every finite subgroup $K \subseteq G$ with $H \subseteq K$ we have $H = K$. In this section we will make the assumption $(M_{\mathcal{T}_R \subseteq \mathcal{F}_{in}})$ appearing in [20, Notation 2.7], i.e., every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup.

Let \mathcal{M} be the set of conjugacy classes of maximal finite subgroups of G . Let $N_G M$ be the normalizer of $M \subseteq G$. Put $W_G M = N_G M / M$. Denote by $p_M: N_G M \rightarrow W_G M$ the canonical projection. Notice that the group $W_G M$ contains no torsion since $M \subseteq G$ is maximal finite. Let $p_M^* E W_G M$ the $N_G M$ -space obtained from the $W_G M$ -space $E W_G M$ by restriction with p_M . Denote by $\underline{E}G$ the classifying space for proper G -action. For more information about these spaces we refer for instance to [3] and [15]. Put $\underline{B}G = G \backslash \underline{E}G$.

We will suppose that G satisfies the Baum-Connes Conjecture (see for example [3] and [18]), i.e., the assembly map

$$(1.1) \quad \text{asmb}: K_i^G(\underline{E}G) \xrightarrow{\cong} K_i(C_r^*(G))$$

is bijective for all $i \in \mathbb{Z}$. Induction with the projection $G \rightarrow \{1\}$ yields a map (see [18, Chapter 6 on pages 732ff])

$$\text{ind}_{G \rightarrow \{1\}}: K_i^G(\underline{E}G) \rightarrow K_i(\underline{B}G).$$

Its composite with the inverse of the assembly map asmb of (1.1) is denoted by

$$\omega_i: K_i(C_r^*(G)) \rightarrow K_i(\underline{B}G).$$

Define

$$\eta_i: K_i(BG) \rightarrow K_i(C_r^*(G))$$

to be the composite

$$K_i(BG) \xrightarrow{(\text{ind}_{G \rightarrow \{1\}})^{-1}} K_i^G(EG) \xrightarrow{K_i^G(f)} K_i^G(\underline{E}G) \xrightarrow{\text{asmb}} K_i(C_r^*(G)),$$

where $f: EG \rightarrow \underline{E}G$ is the up to G -homotopy unique G -map. Let

$$i_M: \ker((p_M)_*: K_i(C_r^*(N_G M)) \rightarrow K_i(C_r^*(W_G M))) \rightarrow K_i(C_r^*(G))$$

be the map induced by the inclusion $N_G M \rightarrow G$. The main result of this section is

Theorem 1.2. *Suppose that G satisfies condition $(M_{\mathcal{T}_R \subseteq \mathcal{F}_{in}})$ appearing in [20, Notation 2.7], i.e., every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup. Assume that the Baum-Connes Conjecture holds for G and for $W_G M$ and $N_G M$ for all $(M) \in \mathcal{M}$.*

(i) *Then there is a long exact sequence*

$$\begin{aligned} \dots &\xrightarrow{\partial_{i+1}} \bigoplus_{(M) \in \mathcal{M}} \ker((p_M)_*: K_i(C_r^*(N_G M)) \rightarrow K_i(C_r^*(W_G M))) \\ &\xrightarrow{\bigoplus_{(M) \in \mathcal{M}} i_M} K_i(C_r^*(G)) \xrightarrow{\omega_i} K_i(\underline{B}G) \\ &\xrightarrow{\partial_i} \bigoplus_{(M) \in \mathcal{M}} \ker((p_M)_*: K_{i-1}(C_r^*(N_G M)) \rightarrow K_{i-1}(C_r^*(W_G M))) \\ &\xrightarrow{\bigoplus_{(M) \in \mathcal{M}} i_{i-1}} K_{i-1}(C_r^*(G)) \xrightarrow{\omega_{i-1}} \dots; \end{aligned}$$

(ii) *For every $i \in \mathbb{Z}$ the map $\omega_i: K_i(C_r^*(G)) \rightarrow K_i(\underline{B}G)$ is split surjective after inverting the orders of all finite subgroups of G .*

For every $i \in \mathbb{Z}$ the homomorphism

$$\eta_i \oplus \bigoplus_{(M) \in \mathcal{M}} i_M : K_i(BG) \oplus \bigoplus_{(M) \in \mathcal{M}} \ker((p_M)_* : K_i(C_r^*(N_G M)) \rightarrow K_i(C_r^*(W_G M))) \rightarrow K_i(C_r^*(G))$$

is bijective after inverting the orders of all finite subgroups of G .

The rest of this section is devoted to the proof of Theorem 1.2

Proof of Theorem 1.2. We have a G -pushout of G -CW-complexes (see [20, Corollary 2.10])

$$(1.3) \quad \begin{array}{ccc} \coprod_{(M) \in \mathcal{M}} G \times_{N_G M} EN_G M & \xrightarrow{i} & EG \\ \downarrow \coprod_{(M) \in \mathcal{M}} \text{id}_G \times_{N_G M} f_M & & \downarrow f \\ \coprod_{(M) \in \mathcal{M}} G \times_{N_G M} p_M^* EW_G M & \xrightarrow{j} & \underline{EG} \end{array}$$

where $f_M : EN_G M \rightarrow p_M^* EW_G M$ is some cellular $N_G M$ -map, $f : EG \rightarrow \underline{EG}$ is some G -map, and i and j are inclusions of G -CW-complexes.

Dividing out the G -action yields a pushout of CW -complexes

$$(1.4) \quad \begin{array}{ccc} \coprod_{(M) \in \mathcal{M}} BN_G M & \longrightarrow & BG \\ \downarrow & & \downarrow f \\ \coprod_{(M) \in \mathcal{M}} BW_G M & \longrightarrow & \underline{BG} \end{array}$$

The associated Mayer-Vietoris sequences yield a commutative diagram whose columns are exact and whose horizontal arrows are given by induction with the projection $G \rightarrow \{1\}$

$$(1.5) \quad \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \bigoplus_{(M) \in \mathcal{M}} K_i^{N_G M}(EN_G M) & \longrightarrow & \bigoplus_{(M) \in \mathcal{M}} K_i(BN_G M) \\ \downarrow & & \downarrow \\ K_i^G(EG) \oplus \bigoplus_{(M) \in \mathcal{M}} K_i^{N_G M}(p_M^* EW_G M) & \longrightarrow & K_i(BG) \oplus \bigoplus_{(M) \in \mathcal{M}} K_i(BW_G M) \\ \downarrow & & \downarrow \\ K_i^G(\underline{EG}) & \longrightarrow & K_i(\underline{BG}) \\ \downarrow & & \downarrow \\ \bigoplus_{(M) \in \mathcal{M}} K_{i-1}^{N_G M}(EN_G M) & \longrightarrow & \bigoplus_{(M) \in \mathcal{M}} K_{i-1}(BN_G M) \\ \downarrow & & \downarrow \\ K_{i-1}^G(EG) \oplus \bigoplus_{(M) \in \mathcal{M}} K_{i-1}^{N_G M}(p_M^* EW_G M) & \longrightarrow & K_{i-1}(BG) \oplus \bigoplus_{(M) \in \mathcal{M}} K_{i-1}(BW_G M) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

Since G acts freely on EG and $N_G M$ acts freely on $EN_G M$, the maps given by induction with the projection to the trivial group

$$\begin{aligned} K_i^G(EG) &\xrightarrow{\cong} K_i(BG); \\ K_i^{N_G M}(EN_G M) &\xrightarrow{\cong} K_i(BN_G M), \end{aligned}$$

are bijective for all $i \in \mathbb{Z}$ and $(M) \in \mathcal{M}$.

Define the map

$$k'_i: \bigoplus_{(M) \in \mathcal{M}} K_i^{N_G M}(p_M^* EW_G M) \rightarrow K_i^G(\underline{EG}) \oplus \bigoplus_{(M) \in \mathcal{M}} K_i(BW_G M)$$

to be the product of the map

$$\bigoplus_{(M) \in \mathcal{M}} K_i^G(u_M) \circ \text{ind}_{N_G M \rightarrow G}: K_i^{N_G M}(p_M^* EW_G M) \rightarrow K_i^G(\underline{EG})$$

for the up to G -homotopy unique G -map $u_M: G \times_{N_G M} p_M^* EW_G M \rightarrow \underline{EG}$ and the map

$$\bigoplus_{(M) \in \mathcal{M}} \text{ind}_{N_G M \rightarrow \{1\}}: \bigoplus_{(M) \in \mathcal{M}} K_i^{N_G M}(p_M^* EW_G M) \rightarrow \bigoplus_{(M) \in \mathcal{M}} K_i(BW_G M).$$

Define the map

$$j'_i: K_i^G(\underline{EG}) \oplus \bigoplus_{(M) \in \mathcal{M}} K_i(BW_G M) \rightarrow K_i(\underline{BG})$$

to be the direct sum of the map

$$\text{ind}_{G \rightarrow \{1\}}: K_i^G(\underline{EG}) \rightarrow K_i(\underline{BG})$$

and (-1) -times the map

$$\bigoplus_{(M) \in \mathcal{M}} K_i(G \setminus u_M): \bigoplus_{(M) \in \mathcal{M}} K_i(BW_G M) \rightarrow K_i(\underline{BG}).$$

Define the map

$$\partial'_i: K_i(\underline{BG}) \rightarrow \bigoplus_{(M) \in \mathcal{M}} K_{i-1}^{N_G M}(p_M^* EW_G M)$$

to be the composite of the three maps

$$\begin{aligned} K_i(\underline{BG}) &\xrightarrow{\delta_i} \bigoplus_{(M) \in \mathcal{M}} K_{i-1}(BN_G M) \xrightarrow{\bigoplus_{(M) \in \mathcal{M}} (\text{ind}_{N_G M \rightarrow \{1\}})^{-1}} \bigoplus_{(M) \in \mathcal{M}} K_{i-1}^{N_G M}(EN_G M) \\ &\xrightarrow{\bigoplus_{(M) \in \mathcal{M}} K_{i-1}(f_M)} \bigoplus_{(M) \in \mathcal{M}} K_{i-1}^{N_G M}(p_M^* EW_G M), \end{aligned}$$

where δ_i is the boundary map appearing in the right column of the diagram (1.5) and $f_M: EN_G M \rightarrow p_M^* EW_G M$ is the up to $N_G M$ -homotopy unique $N_G M$ -map.

The exact columns in the diagram (1.5) can be spliced together to the following long exact sequence.

$$\begin{aligned} (1.6) \quad \dots &\xrightarrow{\partial'_{i+1}} \bigoplus_{(M) \in \mathcal{M}} K_i^{N_G M}(p_M^* EW_G M) \xrightarrow{k'_i} K_i^G(\underline{EG}) \oplus \bigoplus_{(M) \in \mathcal{M}} K_i(BW_G M) \\ &\xrightarrow{j'_i} K_i(\underline{BG}) \xrightarrow{\partial'_i} \bigoplus_{(M) \in \mathcal{M}} K_{i-1}^{N_G M}(p_M^* EW_G M) \\ &\xrightarrow{k'_{i-1}} K_{i-1}^G(\underline{EG}) \oplus \bigoplus_{(M) \in \mathcal{M}} K_{i-1}(BW_G M) \xrightarrow{j'_{i-1}} \dots \end{aligned}$$

We will need the following lemmas.

Lemma 1.7. *Consider any group G and any $i \in \mathbb{Z}$. Let $f: EG \rightarrow \underline{EG}$ be the up to G -homotopy unique G -map. Denote by $\bar{f}: BG \rightarrow \underline{BG}$ the induced map on the G -quotients. Then the composite*

$$K_i(BG) \xrightarrow{(\text{ind}_{G \rightarrow \{1\}})^{-1}} K_i^G(EG) \xrightarrow{f_*} K_i^G(\underline{EG}) \xrightarrow{\text{ind}_{G \rightarrow \{1\}}} K_i(\underline{BG})$$

agrees with the map

$$\bar{f}_*: K_i(BG) \rightarrow K_i(\underline{BG}).$$

This map is bijective after inverting the order of all finite subgroups of G .

Proof. Because induction is natural, we obtain the commutative diagram

$$\begin{array}{ccc} K_i^G(EG) & \xrightarrow{\text{ind}_{G \rightarrow \{1\}}} & K_i(BG) \\ \downarrow f_* & & \downarrow (G \setminus f)_* = \bar{f}_* \\ K_i^G(\underline{EG}) & \xrightarrow{\text{ind}_{G \rightarrow \{1\}}} & K_i(\underline{BG}) \end{array}$$

The upper horizontal arrow is bijective since G acts freely on EG . It remains to show that the right vertical arrow is bijective after inverting the orders of all finite subgroups of G .

Let Λ be the ring $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ obtained from \mathbb{Z} by inverting the orders of all finite subgroups of G . By a spectral sequence argument it suffices to show that $\bar{f}_*: H_l(BG, \Lambda) \rightarrow H_l(\underline{BG}, \Lambda)$ is bijective for all $l \in \mathbb{Z}$. This follows from the fact that the ΛG -chain map $C_*(f): C_*(EG) \otimes_{\mathbb{Z}} \Lambda \rightarrow C_*(\underline{EG}) \otimes_{\mathbb{Z}} \Lambda$ is a homology equivalence of projective ΛG -chain complexes and hence a ΛG -chain homotopy equivalence. \square

Let $p: G \rightarrow H$ be an epimorphism with finite kernel. Then restriction defines for every H -CW-complex X and every $i \in \mathbb{Z}$ a natural map

$$\text{res}_p(X): K_i^H(X) \rightarrow K_i^G(\text{res}_p(X))$$

where $\text{res}_p(X)$ is the G -CW-complex whose underlying space is X and for which $g \in G$ acts on X by multiplication with $p(g)$. This follows from the methods developed in [18, Chapter 6 on pages 732ff]. The maps $\text{res}_p(X)$ above define a transformation of H -homology theories. Notice that we obtain for every H -CW-complex a natural H -homeomorphism

$$\alpha(X): \text{ind}_p \text{res}_p(X) \xrightarrow{\cong} X$$

that is the adjoint of the identity $\text{id}: \text{res}_p(X) \rightarrow \text{res}_p(X)$. Explicitly it sends $(h, x) \in H \times_G \text{res}_p(X)$ to $h \cdot x$. The inverse sends x to $(1, x)$. We leave it to the reader to check the proof of the next lemma which is just a direct inspection of the definitions.

Lemma 1.8. *Let $p: G \rightarrow H$ be an epimorphism with finite kernel. Then for every H -CW-complex X the composite*

$$K_i^H(X) \xrightarrow{\text{res}_p} K_i^G(\text{res}_p(X)) \xrightarrow{\text{ind}_p} K_i^H(\text{ind}_p \circ \text{res}_p(X)) \xrightarrow{K_i^H(\alpha(X))} K_i^H(X)$$

is the identity.

Lemma 1.8 applied to the projection $N_G M \rightarrow W_G M$ and the $W_G M$ -space $EW_G M$ implies that $\text{ind}_{N_G M \rightarrow W_G M}: K_i^{N_G M}(p_M^* EW_G M) \rightarrow K_i^{W_G M}(EW_G M)$ is split surjective. Since $\text{ind}_{W_G M \rightarrow \{1\}}: K_i^{W_G M}(EW_G M) \rightarrow K_i(BW_G M)$ is bijective, the map

$$\text{ind}_{N_G M \rightarrow \{1\}}: K_i^{N_G M}(p_M^* EW_G M) \rightarrow K_i(BW_G M)$$

is split surjective. Now assertion (i) of Theorem 1.2 follows from the long exact sequence (1.6) and the fact that $EW_G M$ is a model for $\underline{EW}_G M$ and $p_M^* EW_G M$ is a model for $\underline{EN}_G M$ because $W_G M$ is the torsion-free quotient of $N_G M$ by a finite group. We obtain assertion (ii) from assertion (i) and Lemma 1.7. This finishes the proof of Theorem 1.2. \square

Example 1.9 (Groups satisfying property $(M_{\mathcal{T} \subseteq \mathcal{F}_{\text{in}}})$). Consider an extension of groups $1 \rightarrow \mathbb{Z}^r \xrightarrow{j} G \xrightarrow{q} A \rightarrow 1$ such that A is finitely generated abelian. Suppose that $\text{tors}(A)$ acts freely on \mathbb{Z}^r outside the origin $0 \in \mathbb{Z}^r$.

Then G satisfies condition $(M_{\mathcal{T} \subseteq \mathcal{F}_{\text{in}}})$ by the following argument. Put $H = q^{-1}(\text{tors}(A))$. We obtain an exact sequence $1 \rightarrow \mathbb{Z}^r \rightarrow H \rightarrow \text{tors}(A) \rightarrow 1$ such that the conjugation action of the finite group $\text{tors}(A)$ on \mathbb{Z}^r is free outside the origin. Then H satisfies by [19, Lemma 6.3] the condition $(NM_{\mathcal{T} \subseteq \mathcal{F}_{\text{in}}})$ appearing in [20, Notation 2.7], i.e., every non-trivial finite subgroup of H is contained in a unique maximal finite subgroup and $N_H M = M$ holds for any maximal finite subgroup $M \subseteq H$. We also obtain an exact sequence $1 \rightarrow H \rightarrow G \xrightarrow{\bar{q}} A/\text{tors}(A) \rightarrow 1$. Hence any finite subgroup of G belongs to H . This implies that G satisfies $(M_{\mathcal{T} \subseteq \mathcal{F}_{\text{in}}})$.

Consider any maximal finite subgroup $M \subseteq G$. Then $M \subseteq H$ and have $N_G M \cap \mathbb{Z}^r = \{1\}$ and $N_G M \cap H = M$. Hence q induces an isomorphism $N_G M \rightarrow q(N_G M)$. Since A is abelian, $N_G M$ is abelian. We get an exact sequence $1 \rightarrow M \rightarrow N_G M \rightarrow \bar{q}(N_G M) \rightarrow 1$, where $\bar{q}(N_G M)$ is a finitely generated free abelian group. This implies that $N_G M \cong M \times W_G M$ and $W_G M$ is a finitely generated free abelian group. Let $\tilde{R}_{\mathbb{C}}(M)$ be the kernel of the split surjection $R_{\mathbb{C}}(M) \rightarrow R_{\mathbb{C}}(\{1\})$ sending the class of an M -representation V to $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}M} V)$. An easy calculation shows

$$\ker((p_M)_*: K_i(C_r^*(N_G M)) \rightarrow K_i(C_r^*(W_G M))) \cong K_i(W_G M) \otimes_{\mathbb{Z}} \tilde{R}_{\mathbb{C}}(M).$$

In particular $\ker((p_M)_*: K_i(C_r^*(N_G M)) \rightarrow K_i(C_r^*(W_G M)))$ is torsionfree. Now Theorem 1.2 yields for every $m \in \mathbb{Z}$ a short exact sequence

$$0 \rightarrow \bigoplus_{(M) \in \mathcal{M}} K_i(BW_G M) \otimes_{\mathbb{Z}} \tilde{R}_{\mathbb{C}}(M) \rightarrow K_i(C_r^*(G)) \rightarrow K_i(\underline{BG}) \rightarrow 0$$

which splits after inverting $|\text{tors}(A)|$.

A prototype for this example is $G = R \rtimes R^\times$ for an integral domain R such that the underlying abelian group of R is finitely generated free and the abelian group R^\times is finitely generated, where R^\times acts on R by multiplication. The ring of integers in an algebraic number field is an example by Dirichlet's Unit Theorem (see for instance [21, Theorem 7.4 in Chapter I on page 42]).

2. GROUPS SATISFYING CONDITION $(NM_{\mathcal{T} \subseteq \mathcal{F}_{\text{in}}})$

Let G be a group. In this section we will make the assumption $(NM_{\mathcal{T} \subseteq \mathcal{F}_{\text{in}}})$ appearing in [20, Notation 2.7], i.e., every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup and $N_G M = M$ holds for any maximal finite subgroup $M \subseteq G$.

2.1. On the topological K-theory of the group C^* -algebra.

Let $\tilde{R}_{\mathbb{C}}(M)$ be the kernel of the split surjection $R_{\mathbb{C}}(M) \rightarrow R_{\mathbb{C}}(\{1\})$ sending the class of an M -representation V to $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}M} V)$. It corresponds under the identifications $R_{\mathbb{C}}(M) = K_0^M(\text{pt})$ and $R_{\mathbb{C}}(\{1\}) = K_0^{\{1\}}(\text{pt})$ to the homomorphism $\text{ind}_{M \rightarrow \{1\}}: K_0^M(\text{pt}) \rightarrow K_0^{\{1\}}(\text{pt})$. Notice that $K_0(C_r^*(M)) = R_{\mathbb{C}}(M)$ and $\tilde{R}_{\mathbb{C}}(M)$ are finitely generated free abelian groups and $K_1(C_r^*(M)) = 0$ for every finite group M .

The next result is a direct consequence of Theorem 1.2.

Corollary 2.1. *Suppose that G satisfies condition $(NM_{\mathcal{T} \subseteq \mathcal{F}in})$ appearing in [20, Notation 2.7], i.e., every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup and for every maximal finite subgroup M we have $N_G M = M$. Suppose that G satisfies the Baum-Connes Conjecture.*

(i) *We obtain an isomorphism*

$$\omega_1: K_1(C_r^*(G)) \xrightarrow{\cong} K_1(\underline{BG});$$

(ii) *We obtain a short exact sequence*

$$0 \rightarrow \bigoplus_{(M) \in \mathcal{M}} \tilde{R}_{\mathbb{C}}(M) \xrightarrow{\bigoplus_{(M) \in \mathcal{M}} i_M} K_0(C_r^*(G)) \xrightarrow{\omega_0} K_0(\underline{BG}) \rightarrow 0,$$

where the map i_M comes from the inclusion $M \rightarrow G$ and the identification $R_{\mathbb{C}}(M) = K_0(C_r^*(M))$. It splits if one inverts the orders of all finite subgroups of G .

The homomorphisms

$$\begin{aligned} \eta_0 \oplus \bigoplus_{(M) \in \mathcal{M}} i_M: K_0(BG) \oplus \bigoplus_{(M) \in \mathcal{M}} \tilde{R}_{\mathbb{C}}(M) &\rightarrow K_0(C_r^*(G)) \\ \eta_1: K_1(BG) &\rightarrow K_1(C_r^*(G)) \end{aligned}$$

are bijective after inverting the orders of all finite subgroups of G .

3. THE COHOMOLOGY OF Γ AND OF THE ASSOCIATED TOROIDAL ORBIFOLD QUOTIENT

In this section we compute the cohomology of $B\Gamma$ and $\underline{B}\Gamma$ which is also called the associated toroidal orbifold quotient. We need some preliminaries.

Lemma 3.1. *Let $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow \mathbb{Z}/m \rightarrow 1$ be an extension such that the conjugation action of \mathbb{Z}/m on \mathbb{Z}^n is free outside the origin. Then the extension splits, the group Γ is a crystallographic group of rank n and possesses a finite n -dimensional Γ -CW-model for $\underline{E}\Gamma$.*

Proof. Let $\gamma \in \Gamma$ be an element in Γ which is mapped under $\Gamma \rightarrow \mathbb{Z}/m$ to a generator of \mathbb{Z}/m . Then γ^m belongs to \mathbb{Z}^n and $\gamma\gamma^m\gamma^{-1} = \gamma^m$. Since γ is non-trivial and the conjugation action of \mathbb{Z}/m on \mathbb{Z}^n is free outside the origin, γ^m is the origin in \mathbb{Z}^n . This implies $\gamma^m = 1$ in Γ .

The subgroup \mathbb{Z}^n of Γ is normal and its own centralizer in Γ . Hence Γ is a crystallographic group of rank n and has a finite n -dimensional G -CW-model for $\underline{E}\Gamma$, namely \mathbb{R}^n with the associated Γ -action, by [5, Propositions 1.12]. \square

One key ingredient for the sequel is the following result from Langer-Lück [11, Theorem 0.1 and Theorem 0.5].

Theorem 3.2 (Tate cohomology). *Suppose that the \mathbb{Z}/m -action on \mathbb{Z}^n is free outside the origin. Then:*

(i) *We get for the Tate cohomology*

$$\hat{H}^i(\mathbb{Z}/m; \Lambda^j(\mathbb{Z}_{\rho}^n)) = 0$$

for all i, j for which $i + j$ is odd;

(ii) *The Lyndon-Serre sequence associated to the extension $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow \mathbb{Z}/m \rightarrow 1$ collapses in the strongest sense, i.e., all differentials in the E_r -term are trivial for all $r \geq 2$, and all extension problems at the E_{∞} -level are trivial.*

The main result of this section is

Theorem 3.3 (Cohomology of $B\Gamma$ and $\underline{B}\Gamma$). *Consider the extension of groups $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{\pi} \mathbb{Z}/m \rightarrow 1$. Assume that the conjugation action of \mathbb{Z}/m on \mathbb{Z}^n is free outside the origin $0 \in \mathbb{Z}^n$. Put for $i \geq 0$*

$$r_i := \text{rk}_{\mathbb{Z}}((\Lambda^i \mathbb{Z}_{\rho}^n)^{\mathbb{Z}/m}) = \text{rk}_{\mathbb{Z}}(H^i(\mathbb{Z}_{\rho}^n)^{\mathbb{Z}/m}).$$

Then

(i) For $i \geq 0$

$$H^i(\Gamma) \cong \begin{cases} \mathbb{Z}^{r_i} \oplus \bigoplus_{l=0}^{i-1} \hat{H}^{i-l}(\mathbb{Z}/m, \Lambda^l \mathbb{Z}_{\rho}^n) & i \text{ even}; \\ \mathbb{Z}^{r_i} & i \text{ odd}, i \geq 3; \\ 0 & i = 1. \end{cases}$$

(ii) The map induced by the various inclusions

$$\varphi^i: H^i(\Gamma) \rightarrow \bigoplus_{(M) \in \mathcal{M}} H^i(M)$$

is bijective for $i > n$;

(iii) For $i \geq 0$

$$H^i(\underline{B}\Gamma) \cong \begin{cases} \mathbb{Z}^{r_i} & i \text{ even}; \\ \mathbb{Z}^{r_i} \oplus \bigoplus_{l=i}^n \hat{H}^l(\mathbb{Z}/m, \Lambda^l \mathbb{Z}_{\rho}^n) & i \text{ odd}, i \geq 3; \\ 0 & i = 1. \end{cases}$$

Proof. (i) follows directly from Theorem 3.2 since $r_1 = 0$.

(ii) The pushout (1.4) reduces to the following pushout of CW -complexes since Γ satisfies the condition $(NM_{\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{F}_{\text{in}}})$ by [19, Lemma 6.3].

$$(3.4) \quad \begin{array}{ccc} \coprod_{(M) \in \mathcal{M}} BM & \longrightarrow & B\Gamma \\ \downarrow & & \downarrow f \\ \coprod_{(M) \in \mathcal{M}} \text{pt} & \longrightarrow & \underline{B}\Gamma \end{array}$$

Since $H^{2i+1}(M) = 0$ for all i and all $(M) \in \mathcal{M}$, the associated Mayer-Vietoris sequence yields the long exact sequence

$$(3.5) \quad 0 \rightarrow H^{2i}(\underline{B}\Gamma) \xrightarrow{\bar{f}^*} H^{2i}(\Gamma) \xrightarrow{\varphi^{2i}} \bigoplus_{(M) \in \mathcal{M}} \tilde{H}^{2i}(M) \xrightarrow{\delta^{2i}} H^{2i+1}(\underline{B}\Gamma) \xrightarrow{\bar{f}^*} H^{2i+1}(\Gamma) \rightarrow 0$$

where φ^{2i} is the map induced by the various inclusions $M \rightarrow \Gamma$. Since there exists a n -dimensional model for $\underline{B}\Gamma$ by Lemma 3.1, assertion (ii) follows.

(iii) In the sequel let i be an integer with $i \geq 1$. Recall that the Lyndon-Serre spectral sequence associated to the extension $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow \mathbb{Z}/m \rightarrow 1$ yields a descending filtration

$$H^l(\Gamma) = F^{0,l} \supset F^{1,l-1} \supset \dots \supset F^{l,0} \supset F^{l+1,-1} = 0$$

such that $F^{r,l-r}/F^{r+1,l-r-1} \cong E_{\infty}^{r,l-r}$. Let $\beta \in H^2(\mathbb{Z}/m)$ be a fixed generator. Recall that $E_2^{2,0} = H^2(\mathbb{Z}/p; H^0(\mathbb{Z}_{\rho}^n)) = H^2(\mathbb{Z}/p)$ so that we can think of β as an element in $E_2^{2,0}$. We conclude $E_2^{i,j} = E_{\infty}^{i,j}$ from Theorem 3.2 (ii). From the multiplicative structure of the spectral sequence we see that the image of the map

$$- \cup \pi^*(\beta)^n: H^{2i}(\Gamma) \rightarrow H^{2i+2n}(\Gamma)$$

lies in $F^{2n,2i}$ and the following diagram commutes

$$(3.6) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ F^{1,2i-1} & \xrightarrow[-\cup \pi^*(\beta)^n]{\cong} & F^{2n+1,2i-1} \\ \downarrow & & \downarrow \\ H^{2i}(\Gamma) & \xrightarrow[-\cup \pi^*(\beta)^n]{} & F^{2n,2i} \\ \downarrow & & \downarrow \\ E_{\infty}^{0,2i} & \xrightarrow[-\cup \beta^n]{} & E_{\infty}^{2n,2i} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

where the columns are exact.

Next we show that the upper horizontal arrow is bijective. Namely, we prove by induction over $r = -1, 0, 1, \dots, 2i-1$ that the map

$$-\cup \pi^*(\beta)^n: F^{2i-r,r} \rightarrow F^{2i-r+2n,r}$$

is bijective. The induction beginning $r = -1$ is trivial since then both the source and the target are trivial, and the induction step from $r-1$ to r follows from the Five-Lemma and the fact that the map

$$-\cup \beta^n: E_{\infty}^{2i-r,r} = H^{2i-r}(\mathbb{Z}/m; H^r(\mathbb{Z}_{\rho}^n)) \rightarrow E_{\infty}^{2i-r+2n,r} = H^{2i-r+2n}(\mathbb{Z}/m; H^r(\mathbb{Z}_{\rho}^n))$$

is bijective for $1 \leq 2i-r$

The bottom horizontal map in diagram (3.6) can be identified with the composition of the canonical quotient map

$$(\Lambda^{2i}\mathbb{Z}_{\rho}^n)^{\mathbb{Z}/m} = H^0(\mathbb{Z}/m; H^{2i}(\mathbb{Z}_{\rho}^n)) \rightarrow \hat{H}^0(\mathbb{Z}/m; H^{2i}(\mathbb{Z}_{\rho}^n))$$

with the isomorphism

$$-\cup \beta^n: \hat{H}^0(\mathbb{Z}/m; H^{2i}(\mathbb{Z}_{\rho}^n)) \xrightarrow{\cong} \hat{H}^{2n}(\mathbb{Z}/m; H^{2i}(\mathbb{Z}_{\rho}^n)).$$

We conclude from the Snake-Lemma that the middle map in diagram (3.6) is an epimorphism and that its kernel fits into an exact sequence

$$(3.7) \quad 0 \rightarrow \ker(-\cup \pi^*(\beta)^n: H^{2i}(\Gamma) \rightarrow H^{2i+2n}(\Gamma)) \rightarrow (\Lambda^{2i}\mathbb{Z}_{\rho}^n)^{\mathbb{Z}/m} \rightarrow \hat{H}^0(\mathbb{Z}/m; H^{2i}(\mathbb{Z}_{\rho}^n)) \rightarrow 0.$$

We have the following commutative diagram

$$(3.8) \quad \begin{array}{ccc} H^{2i}(\Gamma) & \xrightarrow{\varphi^{2i}} & \bigoplus_{(M) \in \mathcal{M}} H^{2i}(M) \\ \downarrow -\cup \pi^*(\beta)^n & & \downarrow \cong \\ F^{2n,2i} & & \bigoplus_{(M) \in \mathcal{M}} (-\cup (\pi \circ i_M)^* \beta^n) \\ \downarrow \iota & & \downarrow \\ H^{2i+2n}(\Gamma) & \xrightarrow[\cong]{\varphi^{2i+2n}} & \bigoplus_{(M) \in \mathcal{M}} H^{2i+2n}(M) \end{array}$$

where ι is the inclusion, the lower horizontal map is bijective because of (3.5) and the existence of a n -dimensional model for $\underline{E}\Gamma$ by Lemma 3.1, and the left vertical map is bijective since the map $H^2(\mathbb{Z}/m) \rightarrow H^2(M)$ induced by the injection $\pi \circ i_M: M \rightarrow \mathbb{Z}/m$ sends β to a generator.

Since $\widehat{H}^0(\mathbb{Z}/m; H^{2i}(\mathbb{Z}_\rho^n))$ is finite and

$$\ker(- \cup \pi^*(\beta)^n : H^{2i}(\Gamma) \rightarrow H^{2i+2n}(\Gamma)) = \ker(\varphi^{2i}) \cong H^{2i}(\underline{B}\Gamma)$$

by the exact sequence (3.5) and the commutative diagram (3.8) we conclude for $i \geq 1$

$$H^{2i}(\underline{B}\Gamma) \cong \mathbb{Z}^{r_{2i}}.$$

From exact sequence (3.5) and the commutative diagram (3.8) we obtain the exact sequence

$$0 \rightarrow \text{cok}(\iota : F^{2n, 2i} \rightarrow H^{2i+2n}(\Gamma)) \rightarrow H^{2i+1}(\underline{B}\Gamma) \rightarrow H^{2i+1}(\Gamma) \rightarrow 0.$$

Since $H^{2i+1}(\Gamma) \cong \mathbb{Z}^{r_{2i+1}}$, this sequence splits. We conclude from Theorem 3.2 (ii)

$$\begin{aligned} \text{cok}(\iota) &\cong \bigoplus_{l=0}^{2n-1} E_\infty^{l, 2i+2n-l} \\ &\cong \bigoplus_{l=0}^{2n-1} H^l(\mathbb{Z}/m, \Lambda^{2i+2n-l} \mathbb{Z}_\rho^n) \\ &\cong \bigoplus_{l=2i+n}^{2n-1} H^l(\mathbb{Z}/m, \Lambda^{2i+2n-l} \mathbb{Z}_\rho^n) \\ &\cong \bigoplus_{l=2i+1}^n \widehat{H}^{2i+2n-l}(\mathbb{Z}/m, \Lambda^l \mathbb{Z}_\rho^n) \\ &\cong \bigoplus_{l=2i+1}^n \widehat{H}^l(\mathbb{Z}/m, \Lambda^l \mathbb{Z}_\rho^n). \end{aligned}$$

We conclude for $i \geq 1$.

$$H^{2i+1}(\underline{B}\Gamma) \cong \mathbb{Z}^{r_{2i+1}} \oplus \bigoplus_{l=2i+1}^n \widehat{H}^l(\mathbb{Z}/m, \Lambda^l \mathbb{Z}_\rho^n).$$

Obviously $H^0(\underline{B}\Gamma) \cong \mathbb{Z}^{r_0}$ holds. Since $H^1(\Gamma) = 0$ by assertion (i), we conclude $H^1(\underline{B}\Gamma) = 0$ from the exact sequence (3.5). This finishes the proof of Theorem 3.3. \square

For a computation of the cohomology of Γ and $\underline{B}\Gamma$ in the case where m is a prime and the conjugation action of \mathbb{Z}/m on \mathbb{Z}^n is not required to be free outside the origin, we refer to [2] and [1].

Remark 3.9 (Homology). By the universal coefficient theorems one can figure out the homology as well. It is easier to determine the cohomology because of the multiplicative structure coming from the cup product.

4. TOPOLOGICAL K -THEORY OF CLASSIFYING SPACES

4.1. Comparing K^1 for classifying spaces.

For later purpose we prove

Lemma 4.1. *Suppose that the group G satisfies the condition $(NM_{\mathcal{T}r \subseteq \mathcal{F}in})$ appearing in [20, Notation 2.7] and that there exists a finite G -CW-model for $\underline{E}G$.*

Then the canonical map

$$K_G^1(\underline{E}G) \rightarrow K^1(BG)$$

is bijective.

Proof. Let $\overline{R}_{\mathbb{C}}(M)$ be the cokernel of the homomorphism $R_{\mathbb{C}}(\{1\}) \rightarrow R_{\mathbb{C}}(M)$ given by restriction with $M \rightarrow \{1\}$. It corresponds under the identifications $R_{\mathbb{C}}(\{1\}) = K_{\{1\}}^0(\text{pt})$ and $R_{\mathbb{C}}(M) = K_M^0(\text{pt})$ to the induction homomorphism $\text{ind}_{M \rightarrow \{1\}}: K_{\{1\}}^0(\text{pt}) \rightarrow K_M^0(\text{pt})$. (Notice that we are using the notation of [14, Section 1], in [17] this map is called inflation.) Define

$$\iota_M: R_{\mathbb{C}}(M) = K_M^0(\text{pt}) \xrightarrow{K_M^0(\text{pr})} K_M^0(EM) \xrightarrow{\text{ind}_{M \rightarrow \{1\}}^{-1}} K^0(BM).$$

Let $\tilde{K}^0(X)$ be the cokernel of the map $K^0(\text{pt}) \rightarrow K^0(X)$ induced by the projection $X \rightarrow \text{pt}$. The map ι_M induces an homomorphism

$$\tau_M: \overline{R}_{\mathbb{C}}(M) \rightarrow \tilde{K}^0(BM).$$

We obtain from the Mayer-Vietoris sequences for K_G^* and K^* applied to the G -pushout (1.3) and to the pushout (1.4) the commutative diagram with exact rows (compare [7, Proof of Theorem 7.1 on page 30])

$$\begin{array}{ccccccc} \bigoplus_{(M) \in \mathcal{M}} \overline{R}_{\mathbb{C}}(M) & \longrightarrow & K^1(\underline{B}G) & \longrightarrow & K_G^1(\underline{E}G) & \longrightarrow & 0 \\ \bigoplus_{(M) \in \mathcal{M}} \tau_M \downarrow & & \text{id} \downarrow & & \downarrow & & \\ \bigoplus_{(M) \in \mathcal{M}} \tilde{K}^0(BM) & \longrightarrow & K^1(\underline{B}G) & \longrightarrow & K^1(BG) & \longrightarrow & 0 \end{array}$$

By the Five-Lemma it remains to show that the obvious map

$$\ker(K^1(\underline{B}G) \rightarrow K_G^1(\underline{E}G)) \rightarrow \ker(K^1(\underline{B}G) \rightarrow K^1(BG))$$

is surjective. The group $K^1(\underline{B}G)$ is finitely generated since there is a finite GCW -model for $\underline{E}G$. Hence also $K^1(BG)$ is finitely generated. For any non-trivial element of finite order $g \in G$ its centralizer $C_G\langle g \rangle$ is finite because the condition $(NM_{\mathcal{T} \subseteq \mathcal{F}_{\text{in}}})$ is satisfied. Hence the rank of the finitely generated group $K^1(BG)$ is $\sum_{j \in \mathbb{Z}} \dim_{\mathbb{Q}}(H^{2j+1}(BG; \mathbb{Q}))$ by [16, Theorem 0.1]. The rank of the finitely generated group $K^1(\underline{B}G)$ is $\sum_{j \in \mathbb{Z}} \dim_{\mathbb{Q}}(H^{2j+1}(\underline{B}G; \mathbb{Q}))$ since $\underline{B}G$ is finite and there exists a rational Chern character. Since the obvious map $BG \rightarrow \underline{B}G$ induces isomorphisms $H^j(\underline{B}G; \mathbb{Q}) \xrightarrow{\cong} H^j(BG; \mathbb{Q})$ for all $j \in \mathbb{Z}$, the ranks of the finitely generated abelian groups $K^1(BG)$ and $K^1(\underline{B}G)$ agree. Hence the kernel of the epimorphism $K^1(\underline{B}G) \rightarrow K^1(BG)$ is finite. This implies that there is an integer l such that multiplication with l annihilates the kernel. Therefore it remains to show for every integer $l > 0$ that the obvious composite

$$\begin{aligned} \bigoplus_{(M) \in \mathcal{M}} \overline{R}_{\mathbb{C}}(M) & \xrightarrow{\bigoplus_{(M) \in \mathcal{M}} \tau_M} \bigoplus_{(M) \in \mathcal{M}} \tilde{K}^0(BM) \\ & \rightarrow \left(\bigoplus_{(M) \in \mathcal{M}} \tilde{K}^0(BM) \right) / l \cdot \left(\bigoplus_{(M) \in \mathcal{M}} \tilde{K}^0(BM) \right) \end{aligned}$$

is surjective. Obviously it suffices to show for every $(M) \in \mathcal{M}$ that the composite

$$R_{\mathbb{C}}(M) \xrightarrow{\iota_M} K^0(BM) \rightarrow K^0(BM) / l \cdot K^0(BM)$$

is surjective.

Let \mathbb{I}_M be the augmentation ideal, i.e., the kernel of the ring homomorphism $R_{\mathbb{C}}(M) \rightarrow \mathbb{Z}$ sending $[V]$ to $\dim_{\mathbb{C}}(V)$. If $M_p \subseteq M$ is a p -Sylow subgroup, restriction defines a map $\mathbb{I}_M \rightarrow \mathbb{I}_{M_p}$. Let $\mathbb{I}_p(M)$ be the quotient of $\mathbb{I}(M)$ by the kernel of this map. This is independent of the choice of the p -Sylow subgroup since two p -Sylow subgroups of M are conjugate. There is an obvious isomorphism from $\mathbb{I}_p(M) \xrightarrow{\cong} \text{im}(\mathbb{I}_M \rightarrow \mathbb{I}_{M_p})$.

Then there is an isomorphism of abelian groups (see [16, Theorem 0.3])

$$K^0(BM) \xrightarrow{\cong} \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(M) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p$$

The map $R_{\mathbb{C}}(M) \xrightarrow{\iota_M} K^0(BM)$ can be identified under this isomorphism with the obvious composite

$$R_{\mathbb{C}}(M) \xrightarrow{\cong} \mathbb{Z} \times \mathbb{I}_M \rightarrow \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(M) \rightarrow \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(M) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p.$$

The map $\mathbb{Z} \times \mathbb{I}(M) \rightarrow \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(M)$ is surjective by [16, Lemma 3.4]. Hence it suffices to show that the map

$$\mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(M) \rightarrow \left(\mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(M) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p \right) / l \cdot \left(\mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(M) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p \right)$$

is surjective. Since $\mathbb{I}_p(M)$ can be non-trivial only for finitely many primes, namely those which divide n , it suffices to show for every prime p that the canonical map

$$\mathbb{I}_p(M) \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\text{id} \otimes \text{pr}} \mathbb{I}_p(M) \otimes_{\mathbb{Z}} (\widehat{\mathbb{Z}}_p / l \cdot \widehat{\mathbb{Z}}_p)$$

is onto. This follows from the fact that the composite

$$\mathbb{Z} \rightarrow \widehat{\mathbb{Z}}_p \rightarrow \widehat{\mathbb{Z}}_p / l \cdot \widehat{\mathbb{Z}}_p$$

is surjective. \square

4.2. Comparing K^0 for classifying spaces and group C^* -algebras. Throughout this subsection we fix an extension of groups

$$1 \rightarrow \mathbb{Z}^n \rightarrow G \rightarrow F \rightarrow 1$$

such that F is a finite group.

We prove some general results in this setting under assumption that the conjugation action of F on \mathbb{Z}^n is free outside the origin $0 \in \mathbb{Z}^n$.

Theorem 4.2. *Suppose that the conjugation action of F on \mathbb{Z}^n is free outside the origin $0 \in \mathbb{Z}^n$. Then*

(i) *We obtain an isomorphism*

$$\omega_1: K_1(C_r^*(G)) \xrightarrow{\cong} K_1(\underline{B}G);$$

(ii) *We obtain a short exact sequence*

$$0 \rightarrow \bigoplus_{(M) \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \xrightarrow{\bigoplus_{(M) \in \mathcal{M}} \iota_M} K_0(C_r^*(G)) \xrightarrow{\omega_0} K_0(\underline{B}G) \rightarrow 0.$$

It splits if we invert $|F|$.

Consider the map

$$\overline{k}_* \oplus \bigoplus_{(M) \in \mathcal{M}} \iota_M: \mathbb{Z} \otimes_{\mathbb{Z}F} K_0(C_r^*(\mathbb{Z}^n)) \oplus \bigoplus_{M \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \rightarrow K_0(C_r^*(G)),$$

where \overline{k}_ is the homomorphism induced by the inclusion $k: \mathbb{Z}^n \rightarrow G$. It becomes a bijection after inverting $|F|$.*

Proof. The group G satisfies by [19, Lemma 6.3] the condition $(NM_{\mathcal{T} \subseteq \mathcal{F}_{\text{in}}})$ appearing in [20, Notation 2.7]. Hence the claim follows from Corollary 2.1 except that the map

$$\overline{k}_* \oplus \bigoplus_{(M) \in \mathcal{M}} \iota_M: \mathbb{Z} \otimes_{\mathbb{Z}F} K_0(C_r^*(\mathbb{Z}^n)) \oplus \bigoplus_{(M) \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \rightarrow K_0(C_r^*(G)).$$

becomes bijective after inverting $|F|$. To prove this, it suffices to show that $\omega_0 \circ \overline{k}_*$ is bijective after inverting $|F|$. Using the induction structure on equivariant K -theory (see [13, page 198]) one checks that this map agrees with the composite

$$\mathbb{Z} \otimes_{\mathbb{Z}F} K_0^{\mathbb{Z}^n}(E\mathbb{Z}^n) \xrightarrow{u_* \circ \text{ind}_{\mathbb{Z}^n \rightarrow G}} K_0^G(EG) \xrightarrow{\text{ind}_{G \rightarrow \{1\}}} K_0(BG) \xrightarrow{\overline{f}_*} K_0(\underline{BG})$$

where $u: G \times_{\mathbb{Z}^n} E\mathbb{Z}^n \rightarrow EG$ is the up to G -homotopy unique G -map. Because of Lemma 1.7 it is enough to show that the composite

$$\mathbb{Z} \otimes_{\mathbb{Z}F} K_0^{\mathbb{Z}^n}(E\mathbb{Z}^n) \xrightarrow{u_* \circ \text{ind}_{\mathbb{Z}^n \rightarrow G}} K_0^G(EG) \xrightarrow{\text{ind}_{G \rightarrow \{1\}}} K_0(BG)$$

is bijective after inverting $|F|$. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \otimes_{\mathbb{Z}F} K_0^{\mathbb{Z}^n}(E\mathbb{Z}^n) & \xrightarrow{\text{id}_{\mathbb{Z}} \otimes_{\mathbb{Z}F} \text{ind}_{\mathbb{Z}^n \rightarrow \{1\}}} & \mathbb{Z} \otimes_{\mathbb{Z}F} K_0(B\mathbb{Z}^n) \\ \downarrow u_* \circ \text{ind}_{\mathbb{Z}^n \rightarrow G} & & \downarrow \widehat{k}_* \\ K_0^G(EG) & \xrightarrow{\text{ind}_{G \rightarrow \{1\}}} & K_0(BG) \end{array}$$

where \widehat{k}_* is induced by the inclusion $k: \mathbb{Z}^n \rightarrow G$. The upper horizontal arrow is bijective since \mathbb{Z}^n acts freely on $E\mathbb{Z}^n$. The lower horizontal arrow is bijective since G acts freely on EG . The right vertical arrow is bijective after inverting $|F|$ because of the Leray-Serre spectral sequence applied to the fibration $B\mathbb{Z}^n \rightarrow BG \rightarrow BF$. This finishes the proof of Theorem 4.2. \square

4.3. Topological K -theory of classifying spaces. Throughout this subsection we consider an extension $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow \mathbb{Z}/m \rightarrow 1$ and assume that the \mathbb{Z}/m -action on \mathbb{Z}_ρ^n is free outside the origin. We want to compute $K^1(B\Gamma)$ and $K^0(\underline{B}\Gamma)$. These turn out to be finitely generated free abelian groups.

We will use the *Leray-Serre spectral sequence* for topological K -theory (see [23, Chapter 15]) of the fibration $B\mathbb{Z}^r \rightarrow B\Gamma \rightarrow B\mathbb{Z}/m$ associated to the extension $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow \mathbb{Z}/m \rightarrow 1$. Recall that its E_2 -term is $E_2^{i,j} = H^i(\mathbb{Z}/m; K^j(B\mathbb{Z}^n))$ and it converges to $K^{i+j}(B\Gamma)$ with no lim^1 -term since the inverse system $\{K^l(B\Gamma^{(n)}) \mid n \geq 0\}$ given by the inclusions of skeletons of $B\Gamma$ satisfies the Mittag-Leffler condition because of [17, Theorem 6.5].

Lemma 4.3. *Suppose that the \mathbb{Z}/m -action on \mathbb{Z}^r is free outside the origin. Then*

- (i) *All differentials in the Leray-Serre spectral sequence for topological K -theory associated to the extension are trivial;*
- (ii) *The canonical map*

$$K^1(B\Gamma) \xrightarrow{\cong} K^1(B\mathbb{Z}^n)^{\mathbb{Z}/m}$$

is bijective. In particular

$$K^1(B\Gamma) \cong \mathbb{Z}^{s_1},$$

where

$$s_1 = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} \left(H^{2i+1}(\mathbb{Z}^n; \mathbb{Q})^{\mathbb{Z}/m} \right) = \sum_{i \in \mathbb{Z}} \text{rk}_{\mathbb{Z}} \left((\Lambda^{2i+1} \mathbb{Z}^n)^{\mathbb{Z}/m} \right).$$

Proof. (i) The proof is analogous to the proof of [7, Lemma 3.5 (iii)] but now using Theorem 3.2. instead of [7, Lemma 3.5 (ii)].

(ii) The argument is analogous to the one of the proof of [7, Theorem 3.3 (iii)] using assertion (i) and Theorem 3.2. \square

Lemma 4.4. (i) We have

$$K^0(\underline{B}\Gamma) \cong \mathbb{Z}^{t_0},$$

where

$$t_0 = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} \left(H^{2i}(\mathbb{Z}^r; \mathbb{Q})^{\mathbb{Z}/m} \right) = \sum_{i \in \mathbb{Z}} \text{rk}_{\mathbb{Z}} \left((\Lambda^{2i} \mathbb{Z}^r)^{\mathbb{Z}/m} \right);$$

(ii) We have

$$K_{\Gamma}^0(\underline{E}\Gamma) \cong \mathbb{Z}^{s_0}.$$

where $s_0 = t_0 + (\sum_{(M) \in \mathcal{M}} (|M| - 1))$.

Proof. (i) We first show that $K^0(\underline{B}\Gamma)$ is a finitely generated free abelian group. There is a finite CW -complex model for $\underline{B}\Gamma$ since Γ can be mapped with a finite kernel onto a crystallographic group. This implies that $K^0(\underline{B}\Gamma)$ is finitely generated abelian. Next we show that $K^0(\underline{B}\Gamma)$ is isomorphic to \mathbb{Z}^{t_0} . We explain how one has to modify arguments in [7] to prove this.

The *Atiyah-Hirzebruch spectral sequence* (see [23, Chapter 15]) for topological K -theory

$$E_2^{i,j} = H^i(\underline{B}\Gamma; K^j(\text{pt})) \Rightarrow K^{i+j}(\underline{B}\Gamma)$$

converges since $\underline{B}\Gamma$ has a model which is a finite dimensional CW -complex. Because of the computations of $H^i(\underline{B}\Gamma)$ of Theorem 3.3 (iii), the E^2 -term in the Atiyah-Hirzebruch spectral sequence converging to $K^*(\underline{B}\Gamma)$ looks like

$$E_2^{i,j} \cong \begin{cases} \mathbb{Z}^{\dim_{\mathbb{Q}}(H^i(\mathbb{Z}^n; \mathbb{Q})^{\mathbb{Z}/m})} & i \text{ even}, j \text{ even}; \\ \mathbb{Z}^{\dim_{\mathbb{Q}}(H^i(\mathbb{Z}^n; \mathbb{Q})^{\mathbb{Z}/m})} \oplus A'_i & i \text{ odd}, i \geq 3, j \text{ even}; \\ 0 & i = 1, j \text{ even}; \\ 0 & j \text{ odd}. \end{cases}$$

where each A'_i is a finite abelian group. The argument in the proof of [7, Lemma 3.4] carries over and shows

$$E_{\infty}^{i,j} \cong \begin{cases} \mathbb{Z}^{\dim_{\mathbb{Q}}(H^i(\mathbb{Z}^n; \mathbb{Q})^{\mathbb{Z}/m})} & i \text{ even}, j \text{ even}; \\ \mathbb{Z}^{\dim_{\mathbb{Q}}(H^i(\mathbb{Z}^n; \mathbb{Q})^{\mathbb{Z}/m})} \oplus A_i & i \text{ odd}, i \geq 3, j \text{ even}; \\ 0 & i = 1, j \text{ even}; \\ 0 & j \text{ odd}. \end{cases}$$

where each A_i is a finite abelian group. Now assertion (i) follows by inspecting the Atiyah-Hirzebruch spectral sequence.

(ii) The group Γ satisfies by [19, Lemma 6.3] the condition $(NM_{\mathcal{T}_{\Gamma} \subseteq \mathcal{F}_{\text{in}}})$. Hence the Γ -pushout (1.3) reduces to the Γ -pushout

$$(4.5) \quad \begin{array}{ccc} \coprod_{(M) \in \mathcal{M}} \Gamma \times_{N_{\Gamma} M} EM & \xrightarrow{i} & E\Gamma \\ \downarrow & & \downarrow \\ \coprod_{(M) \in \mathcal{M}} \Gamma/M & \xrightarrow{j} & \underline{E}\Gamma \end{array}$$

We obtain from the Mayer-Vietoris sequences for K_{Γ}^* and K^* applied to the Γ -pushout (4.5) and to the pushout (3.4) analogous to the construction in [7, Lemma 7.2 (i)] the long exact sequence

$$0 \rightarrow K^0(\underline{B}\Gamma) \rightarrow K_{\Gamma}^0(\underline{E}\Gamma) \rightarrow \bigoplus_{(M) \in \mathcal{M}} \overline{R}_{\mathbb{C}}(M) \rightarrow K^1(\underline{B}\Gamma) \rightarrow K_{\Gamma}^1(\underline{E}\Gamma) \rightarrow 0,$$

where $\overline{R}_{\mathbb{C}}(M)$ is the cokernel of the homomorphism $R_{\mathbb{C}}(\{1\}) \rightarrow R_{\mathbb{C}}(M)$ given by restriction with $M \rightarrow \{1\}$. We have already checked in the proof of Lemma 4.1

that the kernel of the map $K^1(\underline{B}\Gamma) \rightarrow K_1^1(\underline{E}\Gamma)$ is finite. Since $\overline{R}_C(M) \cong \mathbb{Z}^{|M|-1}$, the claim follows from assertion (i). \square

Define for $i \in \mathbb{Z}$

$$s_i = \begin{cases} (\sum_{(M) \in \mathcal{M}} (|M| - 1)) + \sum_{l \in \mathbb{Z}} \text{rk}_{\mathbb{Z}}((\Lambda^{2l}\mathbb{Z}^n)^{\mathbb{Z}/m}) & \text{if } i \text{ even;} \\ \sum_{l \in \mathbb{Z}} \text{rk}_{\mathbb{Z}}((\Lambda^{2l+1}\mathbb{Z}^n)^{\mathbb{Z}/m}) & \text{if } i \text{ odd.} \end{cases}$$

Lemma 4.6. *Consider $m \in \mathbb{Z}$. Then*

- (i) *The abelian group $K_i(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}$ is finitely generated free of rank $\sum_{l \in \mathbb{Z}} \text{rk}_{\mathbb{Z}}((\Lambda^{2l+i}\mathbb{Z}^n)^{\mathbb{Z}/m})$;*
- (ii) *The abelian group $K_i^{\Gamma}(\underline{E}\Gamma)$ is finitely generated free of rank s_i ;*
- (iii) *The abelian group $K_i^{\Gamma}(\underline{E}\Gamma)$ is finitely generated free of rank s_i ;*
- (iv) *The abelian group $K_i^{\Gamma}(C_r^*(\Gamma))$ is finitely generated free of rank s_i .*

Proof. (i) Obviously $K_i(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}$ is finitely generated free since $K_i(C_r^*(\mathbb{Z}^n))$ is finitely generated free. By the Baum-Connes Conjecture $K_i(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m} \cong K_i(B\mathbb{Z}^n)^{\mathbb{Z}/m}$. For every CW-complex X , e.g., $X = B\mathbb{Z}^n$, the Chern character yields a natural isomorphism

$$\bigoplus_{k \in \mathbb{Z}} H_{i+2k}(X; \mathbb{Q}) \xrightarrow{\cong} K_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Now the claim follows from the isomorphisms

$$H_i(B\mathbb{Z}^n; \mathbb{Q})^{\mathbb{Z}/m} \cong H^i(B\mathbb{Z}^n; \mathbb{Q})^{\mathbb{Z}/m} \cong ((\Lambda^i \mathbb{Z}^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{\mathbb{Z}/m}.$$

- (ii) This follows from Lemma 4.1, Lemma 4.3 and Lemma 4.4.
- (iii) This follows from assertion (ii) and the universal coefficient theorem for equivariant K -theory as explained in [7, Theorem 8.3 (ii)].
- (iv) This follows from assertion (iii) and the Baum-Connes Conjecture which holds for Γ (see [10]). \square

Now we can give the proof of Theorem 0.1.

Proof of Theorem 0.1.

- (i) The map $\omega_1: K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(\underline{B}\Gamma)$ is bijective by Theorem 4.2 (i). The composite

$$\mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \xrightarrow{k_*} K_1(C_r^*(\Gamma)) \xrightarrow{k^*} K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}$$

is multiplication with the norm element $N_{\mathbb{Z}/m}$ by the Double Coset Formula (see [9, Lemma 3.2]). The kernel of this map is $\widehat{H}^{-1}(\mathbb{Z}/m; K_1(C_r^*(\mathbb{Z}^n)))$ and its cokernel of this map is $\widehat{H}^0(\mathbb{Z}/m; K_1(C_r^*(\mathbb{Z}^n)))$. The group $\widehat{H}^0(\mathbb{Z}/m; K_1(C_r^*(\mathbb{Z}^n)))$ vanishes by Theorem 3.2 and $\widehat{H}^1(\mathbb{Z}/m; K_1(C_r^*(\mathbb{Z}^n)))$ is annihilated by m . Hence the composite above and therefore $k^*: K_1(C_r^*(\Gamma)) \rightarrow K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}$ are surjective. The abelian group $K_1(C_r^*(\Gamma))$ is a finitely generated free abelian group by Lemma 4.6. Obviously this is also true for $K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}$. They have the same rank by Lemma 4.6. Hence the epimorphism $k^*: K_1(C_r^*(\Gamma)) \rightarrow K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}$ must be bijective. This implies that the map $k_*: \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \rightarrow K_1(C_r^*(\Gamma))$ is surjective and its kernel is $\widehat{H}^{-1}(\mathbb{Z}/m; K_1(C_r^*(\mathbb{Z}^n)))$.

- (ii) The composite

$$\overline{k^*} \circ \overline{k_*}: \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} \widetilde{K}_0(C_r^*(\mathbb{Z}^n)) \rightarrow \widetilde{K}_0(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}$$

is given by multiplication with the norm element $N_{\mathbb{Z}/m}$ of \mathbb{Z}/m by the Double Coset Formula (see [9, Lemma 3.2]). Since $\widehat{H}^{-1}(\mathbb{Z}/m, K_0(C_r^*(\mathbb{Z}^n)))$ vanishes by Theorem 3.2, we conclude that it is injective. Hence $\mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} \widetilde{K}_0(C_r^*(\mathbb{Z}^n))$ is

finitely generated free. Now apply Theorem 4.2 (ii).

(iii) This follows from Lemma 4.6 (iv).

(iv) Since m is even, $\mathbb{Z}/2$ is a subgroup of \mathbb{Z}/m . Since the \mathbb{Z}/m -action on \mathbb{Z}^n is free outside the origin, the $\mathbb{Z}/2$ -action on \mathbb{Z}^n must be given by $-\text{id}$. Hence the induced $\mathbb{Z}/2$ -action on $\Lambda^i \mathbb{Z}^n$ is given by $-\text{id}$ for odd i . Hence $(\Lambda^i \mathbb{Z}^n)^{\mathbb{Z}/m}$ vanishes for odd i . This implies $s_1 = 0$. Now the claim follows from assertion (iii). This finishes the proof of Theorem 0.1 \square

5. CONJUGACY CLASSES OF FINITE SUBGROUPS AND COHOMOLOGY

Consider the group extension $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{\pi} \mathbb{Z}/m \rightarrow 1$ as it appears in Theorem 0.1. In particular we assume that the conjugation action of \mathbb{Z}/m on \mathbb{Z}^n is free outside the origin. For the remainder of this paper we will abbreviate $L = \mathbb{Z}^n$ and $G = \mathbb{Z}/m$.

Lemma 5.1.

- (i) Let $H \subseteq \Gamma$ be a non-trivial finite subgroup. Then there is a subgroup $H_{\max} \subseteq \Gamma$ which is uniquely determined by the properties that $H \subseteq H_{\max}$ and H_{\max} is a maximal finite subgroup, i.e., for every finite subgroup $K \subseteq \Gamma$ with $H_{\max} \subseteq K$ we have $K = H_{\max}$;
- (ii) If H is a maximal finite subgroup of Γ , then $N_{\Gamma}H = H$;
- (iii) If $H \subseteq \Gamma$ is a non-trivial finite subgroup of Γ , then $H_{\max} = N_{\Gamma}H$.

Proof. Assertions (i) and (ii) follow from [19, Lemma 6.3]. They imply assertion (iii) since the action of G on L is free outside the origin and therefore $N_{\Gamma}H \cap L = \{0\}$ and hence $|N_{\Gamma}H| < \infty$. \square

Notation 5.2. Define for a subgroup $C \subseteq G$

$$\mathcal{C}(C) := \{(H) \mid H \subseteq \Gamma, \pi(H) = C\},$$

where (H) denotes the conjugacy class of the subgroup $H \subseteq \Gamma$ within Γ . Define

$$\mathcal{M}(C) := \{(H) \in \mathcal{C}(C) \mid H \text{ is a maximal finite subgroup of } \Gamma\}.$$

Put

$$\mathcal{M} := \{(H) \mid H \text{ is a maximal finite subgroup of } \Gamma\}.$$

We have $\mathcal{M} = \coprod_{\{1\} \subsetneq C \subseteq G} \mathcal{M}(C)$.

If $K \subseteq G$ is a subgroup, then we will abbreviate $H^1(K; \text{res}_G^H L)$ by $H^1(K; L)$ when it is clear from the context to which subgroup the G -action on L is restricted.

Lemma 5.3. Let t be a generator $t \in G$. Consider a non-trivial subgroup $C \subseteq G$. Put $k = [G : C]$. Then t^k is a generator of C and there are bijections

$$H^1(C; L)/G \xrightarrow{\cong} \text{cok}((t^k - 1): L \rightarrow L)/G \xrightarrow{\cong} \mathcal{C}(C),$$

where the G -operation on the first and second term above comes from the G -action on L .

Proof. By definition $H^1(C; L)$ is the quotient of the kernel of the map $N: L \rightarrow L$ coming from multiplication with the norm element $N_C = \sum_{g \in C} g$ by the image of the map $(t^k - 1): L \rightarrow L$. For any $l \in L$ the element $N_C \cdot l$ lies in L^C . Since the G -action on L is free outside the origin and C is non-trivial, $L^C = \{0\}$. We conclude that $N_C: L \rightarrow L$ is the trivial map. This explains the first isomorphism.

Fix an element $\gamma \in \Gamma$ of order $m = |\Gamma|$ with $\pi(\gamma) = t$. The second map sends the class of l to the conjugacy class of the subgroup $\langle \gamma^{-k} \cdot l \rangle$ of Γ generated by $\gamma^{-k} \cdot l$. We have to show that it is well-defined. We compute $(\gamma^{-k} \cdot l)^{|C|} = \gamma^{-k \cdot |C|} \cdot (N_C \cdot l) = \gamma^{-|G|} = 1$. Hence $\langle \gamma^{-k} \cdot l \rangle$ is a finite subgroup of Γ whose image under the projection

$\pi: \Gamma \rightarrow G$ is C . Now suppose for two elements $l_0, l_1 \in L$ that they yield the same class in $\text{cok}((t^k - 1): L \rightarrow L)/G$. Then there exists $j \in \mathbb{Z}/m$ and $l' \in L$ with $l_1 = t^j \cdot l_0 + (t^k - 1) \cdot l'$. Put $l'' = t^{-j} \cdot l'$. Then $t^j \cdot ((t^k - 1) \cdot l'' + l_0) = l_1$ and hence $(\gamma^j \cdot l'')(\gamma^{-k} l_0)(\gamma^j \cdot l'')^{-1} = \gamma^{-k} l_1$. This implies that $\langle \gamma^{-k} l_0 \rangle$ and $\langle \gamma^{-k} l_1 \rangle$ are conjugated in Γ .

The second map is surjective since any finite subgroup $H \subseteq \Gamma$ with $\pi(H) = C$ can be written $\langle \gamma^{-k} \cdot l \rangle$ for an appropriate element $l \in L$.

Finally we prove injectivity. Notice for the sequel that any element in Γ can be written uniquely in the form $\gamma^j \cdot l'$ for some $j \in \mathbb{Z}/m$ and $l' \in L$. Suppose that $\langle \gamma^{-k} l_0 \rangle$ and $\langle \gamma^{-k} l_1 \rangle$ are conjugated in Γ for $l_0, l_1 \in L$. Then there exists a natural number $i \in \mathbb{Z}/|C|^\times$ such that for appropriate $j \in \mathbb{Z}/m$ and $l' \in L$ we have $(\gamma^j \cdot l')(\gamma^{-k} l_0)(\gamma^j \cdot l')^{-1} = (\gamma^{-k} l_1)^i$. Applying π to this equation, yields $t^{-k} = t^{-ki}$ and hence $i = 1 \pmod{|C|}$. Hence get $(\gamma^j \cdot l')(\gamma^{-k} l_0)(\gamma^j \cdot l')^{-1} = \gamma^{-k} l_1$. We conclude $\gamma^j \cdot (\gamma^k l' \gamma^{-k} l_0 (l')^{-1}) \gamma^{-j} = l_1$. This yields, when we use in L the additive notation, $t^j \cdot ((t^k - 1)l' + l_0) = l_1$. If we put $l'' = t^j \cdot l'$, we conclude

$$l_1 = t^j \cdot l_0 + (t^k - 1) \cdot l''.$$

This means that l_0 and l_1 define the same class in $\text{cok}((t^k - 1): L \rightarrow L)/G$. \square

Let $C \subseteq D \subseteq G$ be subgroups of G . Since $\pi^{-1}(C) \subseteq \Gamma$ is normal in Γ , we can define a map

$$(5.4) \quad i_{C \subseteq D}: \mathcal{C}(D) \rightarrow \mathcal{C}(C) \quad (H) \mapsto (H \cap \pi^{-1}(C)).$$

Lemma 5.5. *The map $i_{C \subseteq D}: \mathcal{C}(D) \rightarrow \mathcal{C}(C)$ is injective if C is non-trivial.*

Proof. Consider (H_1) and (H_2) such that $i_{C \subseteq D}((H_1)) = i_{C \subseteq D}((H_2))$ holds. This means $(H_1 \cap \pi^{-1}(C)) = (H_2 \cap \pi^{-1}(C))$. We can choose the representatives H_1 and H_2 such that $H_1 \cap \pi^{-1}(C) = H_2 \cap \pi^{-1}(C)$. Since π maps $H_1 \cap \pi^{-1}(C)$ onto C and C is non-trivial, $H_1 \cap \pi^{-1}(C)$ is non-trivial. Let K be the maximal finite subgroup uniquely determined by the property $H_1 \cap \pi^{-1}(C) \subseteq K$. The existence of K and the fact that K contains both H_1 and H_2 as subgroups follows from Lemma 5.1 (i). Hence $H_1 = K \cap \pi^{-1}(D) = H_2$. This implies $(H_1) = (H_2)$. \square

Notation 5.6. For subgroups $C \subseteq D \subseteq G$ put

$$\mathcal{C}(C; D) = \text{im}(i_{C \subseteq D}: \mathcal{C}(D) \rightarrow \mathcal{C}(C))$$

Lemma 5.7. *Let C, D_1, D_2 be subgroups of G with $C \neq \{1\}$, $C \subseteq D_1$ and $C \subseteq D_2$. Then*

$$\mathcal{C}(C; D_1) \cap \mathcal{C}(C; D_2) = \mathcal{C}(C; \langle D_1, D_2 \rangle).$$

Proof. Consider $(H) \in \mathcal{C}(C; D_1) \cap \mathcal{C}(C; D_2)$. Choose $(H_1) \in \mathcal{C}(D_1)$ and $(H_2) \in \mathcal{C}(D_2)$ with $(H_1 \cap \pi^{-1}(C)) = (H_2 \cap \pi^{-1}(C)) = (H)$. We can choose the representatives H, H_1 and H_2 such that $H_1 \cap \pi^{-1}(C) = H_2 \cap \pi^{-1}(C) = H$. The group H is non-trivial because of $\pi(H) = C$. We conclude from Lemma 5.1 (i) that there exists a maximal finite subgroup K which contains H, H_1 and H_2 as subgroup. In particular K contains $\langle H_1, H_2 \rangle$. Hence $\langle H_1, H_2 \rangle$ is a finite subgroup of Γ . It satisfies $\pi(\langle H_1, H_2 \rangle) = \langle D_1, D_2 \rangle$. This implies that $\langle H_1, H_2 \rangle \cap \pi^{-1}(C) = H$ and hence that (H) belongs to $\mathcal{C}(C; \langle D_1, D_2 \rangle)$. We conclude $\mathcal{C}(C; D_1) \cap \mathcal{C}(C; D_2) \subseteq \mathcal{C}(C; \langle D_1, D_2 \rangle)$. Since $\mathcal{C}(C; D_1) \cap \mathcal{C}(C; D_2) \supseteq \mathcal{C}(C; \langle D_1, D_2 \rangle)$ is obviously true, Lemma 5.7 follows. \square

Theorem 5.8 (The order of $\mathcal{C}(M)$). *Let $C \subseteq G$ be a non-trivial subgroup. Let p_1, p_2, \dots, p_s be the prime numbers dividing $[G : C]$ for $i = 1, 2, \dots, s$. Denote by $D_i \subseteq G$ the subgroup such that $C \subseteq D$ and $[D : C] = p_i$. For a non-trivial*

subset $I \subseteq \{1, 2, \dots, s\}$ denote by D_I the subgroup of G uniquely determined by $[D : C] = \prod_{i \in I} p_i$. Then

- (i) $\mathcal{M}(C) = \mathcal{C}(C) \setminus (\bigcup_{i=1}^s \mathcal{C}(C; D_i))$;
- (ii) $|\mathcal{M}(C)| = |\mathcal{C}(C)| + \sum_{I \subseteq \{1, 2, \dots, s\}, I \neq \emptyset} (-1)^{|I|} \cdot |\mathcal{C}(D_I)|$.
- (iii) We have

$$|\mathcal{C}(C)| = |H^1(C; L)/(G/C)| = |\text{cok}((t^k - 1): L \rightarrow L)/(G/C)|,$$

where t is a generator of G , $k = [G : C]$ and the G/C -action comes from the G -action on L ;

- (iv) Put $k = [G : C]$. For $I \subseteq \{1, 2, \dots, s\}, I \neq \emptyset$ define $k_I := \frac{k}{\prod_{i \in I} p_i}$. Then

$$\begin{aligned} |\mathcal{M}(C)| &= |H^1(C; L)/(G/C)| \\ &\quad + \sum_{I \subseteq \{1, 2, \dots, s\}, I \neq \emptyset} (-1)^{|I|} \cdot |H^1(D_I; L)/(G/D_I)| \\ &= |\text{cok}((t^k - 1): L \rightarrow L)/(G/C)| \\ &\quad + \sum_{I \subseteq \{1, 2, \dots, s\}, I \neq \emptyset} (-1)^{|I|} \cdot |\text{cok}((t^{k_I} - 1): L \rightarrow L)/(G/D_I)|. \end{aligned}$$

Proof. (i) If $D \subseteq G$ is any subgroup with $C \subsetneq D$, then there is $i \in \{1, 2, \dots, s\}$ with $C \subseteq D_i \subseteq D$. Since $\mathcal{M}(C)$ is the complement in $\mathcal{C}(C)$ of the union of the subsets $\mathcal{C}(C; D)$ for all D with $C \subsetneq D$, assertion (i) follows.

(ii) Obviously

$$|\mathcal{M}(C)| = |\mathcal{C}(C)| - \left| \bigcup_{i=1}^s \mathcal{C}(C; D_i) \right|.$$

By the classical Inclusion-Exclusion Principle we get

$$\left| \bigcup_{i=1}^s \mathcal{C}(C; D_i) \right| = \sum_{I \subseteq \{1, 2, \dots, s\}, I \neq \emptyset} (-1)^{|I|} \cdot \left| \bigcap_{i \in I} \mathcal{C}(C; D_i) \right|.$$

Since $\bigcap_{i \in I} \mathcal{C}(C; D_i) = \mathcal{C}(C; D_I)$ by Lemma 5.7 and $\mathcal{C}(C; D_I) = \mathcal{C}(D_I)$ by Lemma 5.5, assertion (ii) follows.

(iii) This is a direct consequence of Lemma 5.3.

(iv) This follows directly from assertions (ii) and (iii). \square

Remark 5.9. Theorem 5.8 gives an explicit formula how one can determine $|\mathcal{M}(C)|$ for all subgroups $C \subseteq G$ if one knows the numbers

$$|H^1(C; L)/(G/C)| = |\text{cok}((t^{[G:C]} - 1): L \rightarrow L)/(G/C)|$$

for all subgroups $C \subseteq G$. One can even identify $\mathcal{M}(C)$ for all subgroups $C \subseteq G$ if one knows the sets

$$H^1(C; L)/(G/C) = \text{cok}((t^{[G:C]} - 1): L \rightarrow L)/(G/C)$$

for all subgroups $C \subseteq G$.

6. THE PRIME POWER CASE

The situation simplifies if one considers the case where $m = |G|$ is a prime power. We conclude from Theorem 5.8:

Example 6.1. Suppose that $|G| = p^r$ for some prime p and some natural number r . Let $G_j \subseteq G$ be the subgroup of order p^j for $j = 1, 2, \dots, r$. Then

$$|\mathcal{M}(G_j)| = \begin{cases} |\mathcal{C}(G_j)| - |\mathcal{C}(G_{j+1})| & \text{if } 1 \leq j < r; \\ |\mathcal{C}(G)| & \text{if } j = r, \end{cases}$$

and we have

$$|\mathcal{C}(G_j)| = |H^1(G_j; L)/(G/G_j)| = \left| \text{cok}((t^{p^{r-j}} - 1): L \rightarrow L)/(G/G_j) \right|.$$

Lemma 6.2. *Suppose that $m = |G|$ is a prime power p^r . Put $\zeta = \exp(2\pi i/p^r)$. Let $G_j \subseteq G$ be the subgroup of order p^j for $j = 1, 2, \dots, r$.*

(i) *There is an isomorphism*

$$H^1(G_j; \mathbb{Z}[\zeta]) \cong \mathbb{Z}/p[G/G_j],$$

which is compatible with the G/G_j -action on the source coming from the G -action on $L = \mathbb{Z}[\zeta]$ and the G/G_j -permutation action on the target;

(ii) *We get for all $j = 1, 2, \dots, r$ and natural numbers k*

$$|H^1(G_j; \mathbb{Z}[\zeta]^k)/(G/G_j)| = |\mathbb{Z}/p[G/G_j]^k/(G/G_j)|,$$

where $\mathbb{Z}[\zeta]^k$ and $\mathbb{Z}/p[G/G_j]^k$ denote the k -fold direct sum, or, equivalently, k -fold direct product.

Proof. (i) There is an exact sequence of $\mathbb{Z}G$ -modules $0 \rightarrow \mathbb{Z}[G/G_1] \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z}[\zeta] \rightarrow 0$ which comes from the G -isomorphism $\mathbb{Z}G/T \cdot \mathbb{Z}G \cong \mathbb{Z}[\zeta]$, where $t \in G$ is a generator and $T = 1 + t^{p^{r-1}} + t^{2p^{r-1}} + \dots + t^{(p-1)p^{r-1}} \in \mathbb{Z}G$. The associated long exact cohomology sequence induces an isomorphism, compatible with the G/G_j -actions coming from the G -actions on $\mathbb{Z}[G/G_1]$ and $\mathbb{Z}[\zeta]$,

$$H^1(G_j; \mathbb{Z}[\zeta]) \cong H^2(G_j; \mathbb{Z}[G/G_1]).$$

Let $q: G/G_1 \rightarrow G/G_j$ be the projection. Fix a map of sets $\sigma: G/G_j \rightarrow G/G_1$ such that eG_j is sent to eG_1 and $q \circ \sigma = \text{id}$. Next we define to one another inverse G_j/G_1 -maps

$$\begin{aligned} \phi: G/G_1 &\xrightarrow{\cong} \coprod_{G/G_j} G_j/G_1; \\ \psi: \coprod_{G/G_j} G_j/G_1 &\xrightarrow{\cong} G/G_1. \end{aligned}$$

The map ϕ sends gG_1 to the element $g \cdot \sigma \circ q(gG_1)^{-1} \in G_j/G_1$ in the summand belonging to $q(gG_1)$. The map ψ sends the element $uG_1 \in G_j/G_1$ in the summand belonging to gG_j to $u\sigma(gG_j)$. There is an obvious G -action on G/G_1 . There is precisely one G -action on $\coprod_{G/G_j} G_j/G_1$ for which ϕ and ψ become G -maps. Namely, given $g_0 \in G$ and an element $uG_1 \in G_j/G_1$ in the summand belonging to gG_j , the action of g_0 on $uG_1 \in G_j/G_1$ yields the element $u_0 \cdot uG_1$ in the summand belonging to g_0gG_j for the element $u_0 = g_0 \cdot \sigma(gG_j) \cdot \sigma(q(g_0) \cdot gC_{p^j})^{-1}$ in G_j . The map $H^2(G_j, \mathbb{Z}[G_j/G_1]) \rightarrow H^2(G_j, \mathbb{Z}[G_j/G_1])$ induced by multiplication with u_0 on $\mathbb{Z}[G_j/G_1]$ is the identity since for a projective $\mathbb{Z}[G_j]$ -resolution P_* the $\mathbb{Z}[G_j]$ -chain map $P_* \rightarrow P_*$ given by multiplication with u_0 induces the identity on homology and hence is $\mathbb{Z}[G_j]$ -chain homotopic to the identity. Hence the isomorphism induced by ϕ

$$H^2(G_j; \mathbb{Z}[G/G_1]) \xrightarrow{\cong} \bigoplus_{G/G_j} H^2(G_j; \mathbb{Z}[G_j/G_1])$$

is compatible with the G/G_j -actions if we use on the source the one coming from the G -action on $\mathbb{Z}[G/G_1]$ and on the target the G/G_j -permutation action. By Shapiro's lemma (see [4, (5.2) in VI.5 on page 136])

$$H^2(G_j; \mathbb{Z}[G_j/G_1]) \cong H^2(G_1; \mathbb{Z}) \cong H^2(\mathbb{Z}/p) \cong \mathbb{Z}/p.$$

Now assertion (i) follows.

(ii) We get from assertion (i) bijection of G/G_j -sets

$$H^1(G_j; \mathbb{Z}[\zeta]^k) \cong H^1(G_j; \mathbb{Z}[\zeta])^k \cong \mathbb{Z}/p[G/G_j]^k.$$

It induces a bijection

$$H^1(G_j; \mathbb{Z}[\zeta]^k)/(G/G_j) \xrightarrow{\cong} \mathbb{Z}/p[G/G_j]^k/(G/G_j).$$

□

Lemma 6.3. *Let p be a prime. Put $P = \{0, 1, \dots, (p-1)\}$. Let H be a finite cyclic group of order p^r for $r \geq 1$. Let k be a natural number. Define for a set T*

$$\text{map}(T, P)_{\text{ev}} := \left\{ f: T \rightarrow P \mid \sum_{h \in H} f(h) \equiv 0 \pmod{2} \right\};$$

$$\text{map}(T, P)_{\text{odd}} := \left\{ f: T \rightarrow P \mid \sum_{h \in H} f(h) \equiv 1 \pmod{2} \right\}.$$

Denote by H_j the subgroup of H of order p^j for $j = 0, 1, 2, \dots, r$. Then

(i) We get in the Burnside ring $A(H)$ for $p \neq 2$

$$(\text{map}(H, P)_{\text{ev}} - \text{map}(H, P)_{\text{odd}})^k = [H/H],$$

and for $p = 2$

$$\begin{aligned} & ([\text{map}(H, P)_{\text{ev}}] - [\text{map}(H, P)_{\text{odd}}])^k \\ &= 2^k \cdot [H/H] - 2^{k2^{r-1}-r} \cdot [H] + \sum_{i=1}^{r-1} (2^{k2^{r-i}-r+i} - 2^{k2^{r-i-1}-r+i}) \cdot [H/H_i]; \end{aligned}$$

(ii) We get for all primes p in $A(H)$

$$[\text{map}(H, P)^k] = p^k \cdot [H/H] + \sum_{i=0}^{r-1} (p^{kp^{r-i}-r+i} - p^{kp^{r-i-1}-r+i}) \cdot [H/H_i].$$

Moreover, we have for all primes p the equality of integers

$$|\text{map}(H, P)^k/H| = p^k + \sum_{i=0}^{r-1} (p^{kp^{r-i}-r+i} - p^{kp^{r-i-1}-r+i}).$$

Proof. For $i = 0, 1, 2, \dots, r$ we have the ring homomorphism

$$\text{ch}_i: A(H) \rightarrow \mathbb{Z}, \quad [S] \mapsto |S^{H_i}|.$$

Consider the element in $A(H)$

$$x = \sum_{j=0}^r a_j \cdot [H/H_j].$$

Then we get $\text{ch}_i(x) = \sum_{j=i}^r a_j \cdot p^{r-j}$ for $i = 0, 1, 2, \dots, r$. This implies

$$(6.4) \quad a_j = \begin{cases} \text{ch}_j(x) & j = r; \\ \frac{\text{ch}_j(x) - \text{ch}_{j+1}(x)}{p^{r-j}} & j = 0, 1, 2, \dots, (r-1). \end{cases}$$

One easily checks for a set T by induction over its cardinality

$$(6.5) \quad \text{map}(T, P)_{\text{ev}} = \begin{cases} \frac{p^{|T|}+1}{2} & p \neq 2; \\ 2^{|T|-1} & p = 2; \end{cases}$$

$$(6.6) \quad \text{map}(T, P)_{\text{odd}} = \begin{cases} \frac{p^{|T|}-1}{2} & p \neq 2; \\ 2^{|T|-1} & p = 2. \end{cases}$$

Let $\text{pr}_j: H \rightarrow H/H_j$ be the projection. It induces a bijection

$$(6.7) \quad \text{pr}_j^*: \text{map}(H/H_j, P) \xrightarrow{\cong} \text{map}(H, P)^{H_j}.$$

Given $f: H/H_j \rightarrow P$, we get

$$|H_j| \cdot \sum_{hH_j \in H/H_j} f(hH_j) = \sum_{h \in H} \text{pr}_j^*(f)(h).$$

Hence pr_j^* induces bijections

$$(6.8) \quad \text{map}(H/H_j, P)_{\text{ev}} \cong (\text{map}(H, P)_{\text{ev}})^{H_j};$$

$$(6.9) \quad \text{map}(H/H_j, P)_{\text{odd}} \cong (\text{map}(H, P)_{\text{odd}})^{H_j},$$

provided that $p \neq 2$. If $p = 2$, we get for $j \geq 1$

$$(6.10) \quad (\text{map}(H, P)_{\text{odd}})^{H_j} = \emptyset.$$

We conclude from (6.5) and (6.8) for $p \neq 2$

$$(6.11) \quad \begin{aligned} \text{ch}_j([\text{map}(H, P)_{\text{ev}}]) &= |\text{map}(H, P)_{\text{ev}}^{H_j}| \\ &= |\text{map}(H/H_j, P)_{\text{ev}}| \\ &= \frac{p^{p^{r-j}} + 1}{2}. \end{aligned}$$

Analogously we conclude from (6.6) and (6.9) for $p \neq 2$

$$(6.12) \quad \text{ch}_j([\text{map}(H, P)_{\text{odd}}]) = \frac{p^{p^{r-j}} - 1}{2},$$

We conclude from (6.7) for all primes p

$$(6.13) \quad \text{ch}_j([\text{map}(H, P)^k]) = p^{kp^{r-j}}.$$

We conclude from (6.10) for $p = 2$ for $j \geq 1$

$$(6.14) \quad \text{ch}_j(\text{map}(H, P)_{\text{odd}}) = 0.$$

Now we are ready to prove assertion (i). We conclude from (6.11) and (6.12) for $p \neq 2$ and $j = 0, 1, \dots, r$

$$\begin{aligned} & \text{ch}_j \left(([\text{map}(H, P)_{\text{ev}}] - [\text{map}(H, P)_{\text{odd}}])^k \right) \\ &= (\text{ch}_j([\text{map}(H, P)_{\text{ev}}]) - \text{ch}_j([\text{map}(H, P)_{\text{odd}}]))^k \\ &= \left(\frac{p^{p^{r-j}} + 1}{2} - \frac{p^{p^{r-j}} - 1}{2} \right)^k \\ &= 1. \end{aligned}$$

Hence we obtain from (6.4) for $p \neq 2$ the equation in $A(H)$

$$([\text{map}(H, P)_{\text{ev}}] - [\text{map}(H, P)_{\text{odd}}])^k = [H/H].$$

We conclude from (6.13) and (6.14) for $p = 2$ and $j \geq 1$

$$(6.15) \quad \begin{aligned} & \text{ch}_j \left(([\text{map}(H, P)_{\text{ev}}] - [\text{map}(H, P)_{\text{odd}}])^k \right) \\ &= \text{ch}_j \left(([\text{map}(H, P)] - 2 \cdot [\text{map}(H, P)_{\text{odd}}])^k \right) \\ &= (\text{ch}_j([\text{map}(H, P)]) - 2 \cdot \text{ch}_j([\text{map}(H, P)_{\text{odd}}]))^k \\ &= \text{ch}_j([\text{map}(H, P)^k]) \\ &= 2^{k2^{r-j}}. \end{aligned}$$

For $p = 2$ and $j = 0$ we conclude from (6.5) and (6.6)

$$\begin{aligned}
 (6.16) \quad \text{ch}_0 \left(([\text{map}(H, P)_{\text{ev}}] - [\text{map}(H, P)_{\text{odd}}])^k \right) \\
 &= (|\text{map}(H, P)_{\text{ev}}| - |\text{map}(H, P)_{\text{odd}}|)^k \\
 &= 0^k \\
 &= 0.
 \end{aligned}$$

Equations (6.4), (6.15) and (6.16) imply for $p = 2$

$$\begin{aligned}
 &([\text{map}(H, P)_{\text{ev}}] - [\text{map}(H, P)_{\text{odd}}])^k \\
 &= 2^k \cdot [H/H] - 2^{k2^{r-1}-r} \cdot [H] + \sum_{i=1}^{r-1} (2^{k2^{r-i}-r+i} - 2^{k2^{r-i-1}-r+i}) \cdot [H/H_i].
 \end{aligned}$$

This finishes the proof of assertion (i).

Finally we prove assertion (ii). We conclude from (6.4) and (6.13)

$$\begin{aligned}
 (6.17) \quad [\text{map}(H, P)^k] &= p^k \cdot [H/H] + \sum_{i=0}^{r-1} \frac{p^{kp^{r-i}} - p^{kp^{r-i-1}}}{p^{r-i}} \cdot [H/H_i] \\
 &= p^k \cdot [H/H] + \sum_{i=0}^{r-1} (p^{kp^{r-i}-r+i} - p^{kp^{r-i-1}-r+i}) \cdot [H/H_i].
 \end{aligned}$$

The last equation implies

$$|\text{map}(H, P)^k/H| = p^k + \sum_{i=0}^{r-1} (p^{kp^{r-i}-r+i} - p^{kp^{r-i-1}-r+i}).$$

This finishes the proof of Lemma 6.3. \square

Theorem 6.18 (Prime power case). *Suppose that $m = |G|$ is a prime power p^r for $r \geq 1$. Let $G_j \subseteq G$ be the subgroup of order p^j for $j = 0, 1, 2, \dots, r$.*

Then there is natural number k uniquely determined by $n = p^{r-1} \cdot (p-1) \cdot k$. We obtain for $j = 1, 2, \dots, r$ in $A(G)$

$$[H^1(G_j; L)] = p^k \cdot [G/G] + \sum_{i=0}^{r-j-1} (p^{kp^{r-j-i}-r+j+i} - p^{kp^{r-j-1-i}-r+j+i}) \cdot [G/G_{i+j}],$$

and

$$|\mathcal{M}(G_j)| = \begin{cases} p^{kp^{r-j}-r+j} - p^{kp^{r-j-1}-r+j} & j = 1, 2, \dots, (r-1); \\ p^k & j = r. \end{cases}$$

Proof. We get from Theorem 5.8 (iii),

$$(6.19) \quad |\mathcal{C}(G_j)| = |H^1(G_j; L)/(G/G_j)|.$$

Choose k such that the $\mathbb{Z}_{(p)}[G]$ -module $L_{(p)}$ is isomorphic to $(\mathbb{Z}[\zeta]_{(p)})^k$. Then

$$\begin{aligned}
 n = \text{rk}_{\mathbb{Z}}(L) &= \text{rk}_{\mathbb{Z}_{(p)}}(L_{(p)}) = \text{rk}_{\mathbb{Z}_{(p)}}(\mathbb{Z}[\zeta]_{(p)}^k) = \text{rk}_{\mathbb{Z}}(\mathbb{Z}[\zeta]^k) \\
 &= k \cdot \text{rk}_{\mathbb{Z}}(\mathbb{Z}[\zeta]) = k \cdot p^{r-1} \cdot (p-1).
 \end{aligned}$$

We obtain from [4, Theorem 10.3 in III.10. on page 84] an isomorphism of $\mathbb{Z}[G/G_j]$ -modules

$$H^1(G_j; L) \cong H^1(G_j; \mathbb{Z}[\zeta]^k).$$

Hence we conclude from (6.19)

$$(6.20) \quad |\mathcal{C}(G_j)| = |H^1(G_j; \mathbb{Z}[\zeta]^k)/(G/G_j)|,$$

and we get in $A(G/G_j)$

$$(6.21) \quad [H^1(G_j; L)] = [H^1(G_j; \mathbb{Z}[\zeta]^k)].$$

From Lemma 6.2 (i) and Lemma 6.3 (ii) applied to $H = G/G_j$ we obtain the equation in $A(G/G_j)$ for $j = 1, 2, \dots, r$

$$(6.22) \quad \begin{aligned} [H^1(G_j; \mathbb{Z}[\zeta]^k)] &= [H^1(G_j; \mathbb{Z}[\zeta]^k)] \\ &= [\text{map}(G/G_j, \mathbb{Z}/p)^k] \\ &= p^k \cdot [(G/G_j)/(G/G_j)] \\ &\quad + \sum_{i=0}^{r-j-1} (p^{kp^{r-j-i}-r+j+i} - p^{kp^{r-j-i-1}-r+j+i}) \cdot [(G/G_j)/G_{i+j}/G_j]. \end{aligned}$$

From (6.21) and (6.22) we obtain the following equality in $A(G)$ for $j = 1, 2, \dots, r$

$$[H^1(G_j; L)] = p^k \cdot [G/G] + \sum_{i=0}^{r-j-1} (p^{kp^{r-j-i}-r+j+i} - p^{kp^{r-j-i-1}-r+j+i}) \cdot [G/G_{i+j}].$$

This together with (6.19) implies for $j = 1, 2, \dots, r$

$$|\mathcal{C}(G_j)| = p^k + \sum_{i=0}^{r-j-1} (p^{kp^{r-j-i}-r+j+i} - p^{kp^{r-j-i-1}-r+j+i}).$$

Now the claim follows from Example 6.1 by the following calculation for $j = 1, 2, \dots, (r-1)$

$$\begin{aligned} & p^k + \sum_{i=0}^{r-j-1} (p^{kp^{r-j-i}-r+j+i} - p^{kp^{r-j-i-1}-r+j+i}) \\ & \quad - p^k - \sum_{i=0}^{r-j-2} (p^{kp^{r-j-1-i}+r+j+1+i} - p^{kp^{r-j-1-i-1}-r+j+1+i}) \\ &= \sum_{i=0}^{r-j-1} (p^{kp^{r-j-i}-r+j+i} - p^{kp^{r-j-i-1}-r+j+i}) \\ & \quad - \sum_{i=0}^{r-j-2} (p^{kp^{r-j-1-i}+r+j+1+i} - p^{kp^{r-j-1-i-1}-r+j+1+i}) \\ &= \sum_{i=0}^{r-j-1} (p^{kp^{r-j-i}-r+j+i} - p^{kp^{r-j-i-1}-r+j+i}) \\ & \quad - \sum_{i=1}^{r-j-1} (p^{kp^{r-j-i}+r+j+i} - p^{kp^{r-j-i-1}-r+j+i}) \\ &= p^{kp^{r-j}-r+j} - p^{kp^{r-j-1}-r+j}. \end{aligned}$$

□

Notation 6.23. Given a commutative ring R and an R -module M , denote by $\Lambda^j M = \Lambda_R^j M$ its j -th exterior power. Define R -modules

$$\begin{aligned}\Lambda^{\text{odd}} M &:= \bigoplus_{l \geq 0} \Lambda^{2l+1} M; \\ \Lambda^{\text{ev}} M &:= \bigoplus_{l \geq 0} \Lambda^{2l} M; \\ \Lambda^{\text{all}} M &:= \bigoplus_{j \geq 0} \Lambda^j M.\end{aligned}$$

Define for a rational G -representation V classes in $R_{\mathbb{Q}}(G)$

$$\begin{aligned}[\Lambda^{\text{all}} V] &:= \sum_{j \geq 0} [\Lambda^j V]; \\ [\Lambda^{\text{alt}} V] &:= \sum_{j \geq 0} (-1)^j \cdot [\Lambda^j V];\end{aligned}$$

Lemma 6.24. Suppose that $m = p^r$ for a prime p and an integer $r \geq 1$. Let k be a natural number. Put $\xi = \exp(2\pi i/m)$. Let \mathbb{Q} be the trivial $\mathbb{Q}G$ -module of dimension one.

(i) If $p \neq 2$, we get in $R_{\mathbb{Q}}(G)$

$$[\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] = [\mathbb{Q}] + \frac{2^{k \cdot (p-1) \cdot p^{r-1}} - 1}{p^r} \cdot [\mathbb{Q}G];$$

(ii) Suppose $p = 2$. If $r = 1$, we get in $R_{\mathbb{Q}}(G)$

$$[\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] = 2^{k-1} \cdot [\mathbb{Q}G].$$

If $r \geq 2$, we get in $R_{\mathbb{Q}}(G)$

$$\begin{aligned}[\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] &= 2^k \cdot [\mathbb{Q}] - 2^{k2^{r-2}-r+1} \cdot [\mathbb{Q}[G/G_1]] \\ &\quad + \sum_{i=2}^{r-1} (2^{k2^{r-i}-r+i} - 2^{k2^{r-1-i}-r+i}) \cdot [\mathbb{Q}[G/G_i]] + 2^{k \cdot 2^{r-1}-r} \cdot [\mathbb{Q}G];\end{aligned}$$

(iii) We get in $R_{\mathbb{Q}}(G)$

$$\begin{aligned}[\Lambda^{\text{alt}}(\mathbb{Q}[\zeta]^k)] &= p^k \cdot [\mathbb{Q}] - p^{kp^{r-1}-r} \cdot [\mathbb{Q}G] \\ &\quad + \sum_{i=1}^{r-1} (p^{kp^{r-i}-r+i} - p^{kp^{r-i-1}-r+i}) \cdot [\mathbb{Q}[G/G_i]].\end{aligned}$$

Proof. (i) and (ii) Consider an integer $j \geq 1$. We conclude from [11, (1.5) and Lemma 1.6] (using the notation of this paper) that there exists a sequence of $\mathbb{Q}G$ -modules

$$0 \rightarrow \bigoplus_{\sigma \in S_{<p}} \Gamma^j \mathbb{Q}[G/G_1]_{\sigma} \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_j \rightarrow \Lambda^j \mathbb{Q}[\zeta] \rightarrow 0$$

where each F_l is a finitely generated free $\mathbb{Q}G$ -module and $S_{<p}$ is the subset of S consisting of elements represented by functions $f: \mathbb{Z}/p^{r-1} \rightarrow \{0, 1, \dots, (p-1)\}$ and the G -action on G/G_1 comes from the canonical projection $G \rightarrow G/G_1$. Hence we get for some $a_j \in \mathbb{Z}$ in $R_{\mathbb{Q}}(G)$

$$[\Lambda^j \mathbb{Q}[\zeta]] = (-1)^j \cdot \left[\bigoplus_{\sigma \in S_{<p}} \Gamma^j \mathbb{Q}[G/G_1]_{\sigma} \right] + a_j \cdot [\mathbb{Q}G].$$

Thus we get for an appropriate integer a' in $R_{\mathbb{Q}}(G)$

$$[\Lambda^{\text{all}}\mathbb{Q}[\zeta]] = \left[\bigoplus_{l \geq 0} \bigoplus_{\sigma \in S_{<p}} \Gamma^{2l}\mathbb{Q}[G/G_1]_{\sigma} \right] - \left[\bigoplus_{l \geq 0} \bigoplus_{\sigma \in S_{<p}} \Gamma^{2l+1}\mathbb{Q}[G/G_1]_{\sigma} \right] + a' \cdot [\mathbb{Q}G].$$

Fix a generator $t \in G$. Denote by $[t^s]$ the class of t^s in G/G_1 . A \mathbb{Z} -basis for $\bigoplus_{\sigma \in S_{<p}} \Gamma^j\mathbb{Q}[G/G_1]_{\sigma}$ is given by the set

$$\left\{ [t^0]^{[e_0]} [t^1]^{[e_1]} \dots [t^{p^{r-1}-1}]^{[e_{p^{r-1}-1}]} \mid e_0, e_1, \dots, e_{p^{r-1}-1} \in P, \sum_{i=0}^{p^{r-1}-1} e_i = j \right\}.$$

Hence a \mathbb{Z} -basis for $\bigoplus_{l \geq 0} \bigoplus_{\sigma \in S_{<p}} \Gamma^{2l}\mathbb{Q}[G/G_1]_{\sigma}$ is given by the set

$$\left\{ [t^0]^{[e_0]} [t^1]^{[e_1]} \dots [t^{p^{r-1}-1}]^{[e_{p^{r-1}-1}]} \mid e_0, e_1, \dots, e_{p^{r-1}-1} \in P, \sum_{i=0}^{p^{r-1}-1} e_i \equiv 0 \pmod{2} \right\}.$$

We can identify this set with $\text{map}(G/G_1, P)_{\text{ev}}$ in the obvious way. There is an obvious G/G_1 -action on $\text{map}(G/G_1, P)_{\text{ev}}$ which yields a G -action by the projection $G \rightarrow G/G_1$. Thus we get in $R_{\mathbb{Q}}(G)$

$$\begin{aligned} \left[\bigoplus_{l \geq 0} \bigoplus_{\sigma \in S_{<p}} \Gamma^{2l}\mathbb{Q}[G/G_1]_{\sigma} \right] &= [\mathbb{Q} \text{map}(G/G_1, P)_{\text{ev}}]; \\ \left[\bigoplus_{l \geq 0} \bigoplus_{\sigma \in S_{<p}} \Gamma^{2l+1}\mathbb{Q}[G/G_1]_{\sigma} \right] &= [\mathbb{Q} \text{map}(G/G_1, P)_{\text{odd}}]. \end{aligned}$$

Hence we obtain the following equation in $R_{\mathbb{Q}}(H)$ for some integer a'

$$(6.25) \quad [\Lambda^{\text{all}}\mathbb{Q}[\zeta]] = [\mathbb{Q} \text{map}(G/G_1, P)_{\text{ev}}] - [\mathbb{Q} \text{map}(G/G_1, P)_{\text{odd}}] + a' \cdot [\mathbb{Q}G].$$

By the exponential law we get an isomorphism of $\mathbb{Q}G$ -modules

$$\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k) \cong \bigotimes_{l=1}^k \Lambda^{\text{all}}\mathbb{Q}[\zeta]$$

and hence we get in $R_{\mathbb{Q}}(G)$

$$[\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] = \prod_{l=1}^k [\Lambda^{\text{all}}\mathbb{Q}[\zeta]].$$

For every $\mathbb{Q}G$ -module V the $\mathbb{Q}G$ -module $V \otimes_{\mathbb{Q}} \mathbb{Q}G$ is a free $\mathbb{Q}G$ -module. Hence we obtain in $R_{\mathbb{Q}}(G)$ for an appropriate integer a

$$(6.26) \quad \begin{aligned} [\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] &= ([\mathbb{Q} \text{map}(G/G_1, P)_{\text{ev}}] - [\mathbb{Q} \text{map}(G/G_1, P)_{\text{odd}}])^k + a \cdot [\mathbb{Q}G]. \end{aligned}$$

Next we consider the case $p \neq 2$. Lemma 6.3 (i) and (6.26) imply

$$[\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] = [\mathbb{Q}] + a \cdot [\mathbb{Q}G]$$

Since

$$\begin{aligned} \sum_{j \geq 0} \dim_{\mathbb{Q}} (\Lambda^j \mathbb{Q}[\zeta]^k) &= 2^{\dim_{\mathbb{Q}}(\mathbb{Q}[\zeta]^k)} = 2^{k \cdot (p-1) \cdot p^{r-1}}, \\ \dim_{\mathbb{Q}}(\mathbb{Q}G) &= p^r, \end{aligned}$$

we conclude

$$a = \frac{2^{k \cdot (p-1) \cdot p^{r-1}} - 1}{p^r}$$

This finishes the proof of assertion (i).

Next we treat assertion (ii), i.e., the case $p = 2$. We begin with the case $r = 1$, Then G/G_1 is trivial and (6.26) implies

$$[\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] = a \cdot \mathbb{Q}G.$$

Taking rational dimensions yields $2^k = 2a$ and hence $a = 2^{k-1}$. Hence we get

$$[\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] = 2^{k-1} \cdot [\mathbb{Q}G] \quad \text{if } r = 1.$$

If $r \geq 2$, then G/G_1 is non-trivial and hence Lemma 6.3 (i) and (6.26) imply for some integer a

$$\begin{aligned} (6.27) \quad [\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] &= 2^k \cdot [\mathbb{Q}] - 2^{k2^{r-2}-r+1} \cdot [\mathbb{Q}[G/G_1]] \\ &\quad + \sum_{i=1}^{r-2} (2^{k2^{r-1-i}-r+1+i} - 2^{k2^{r-2-i}-r+1+i}) \cdot [\mathbb{Q}[G/G_{i+1}]] + a \cdot [\mathbb{Q}G] \\ &= 2^k \cdot [\mathbb{Q}] - 2^{k2^{r-2}-r+1} \cdot [\mathbb{Q}[G/G_1]] \\ &\quad + \sum_{i=2}^{r-1} (2^{k2^{r-i}-r+i} - 2^{k2^{r-1-i}-r+i}) \cdot [\mathbb{Q}[G/G_i]] + a \cdot [\mathbb{Q}G]. \end{aligned}$$

If $r = 2$, then taking rational dimensions in (6.27) yields

$$2^{2k} = 2^k - 2^{k-1} \cdot 2 + 2^2 \cdot a$$

and hence $a = 2^{2k-2}$. If $r \geq 3$, taking rational dimension in (6.27) yields

$$\begin{aligned} 2^{k \cdot 2^{r-1}} &= 2^k - 2^{k2^{r-2}-r+1} \cdot 2^{r-1} + \sum_{i=2}^{r-1} (2^{k2^{r-i}-r+i} - 2^{k2^{r-1-i}-r+i}) \cdot 2^{r-i} + a \cdot 2^r \\ &= 2^k - 2^{k2^{r-2}} + \sum_{i=2}^{r-1} (2^{k2^{r-i}} - 2^{k2^{r-1-i}}) + a \cdot 2^r \\ &= 2^k - 2^{k2^{r-2}} + a \cdot 2^r + \sum_{i=2}^{r-1} 2^{k2^{r-i}} - \sum_{i=2}^{r-1} 2^{k2^{r-1-i}} \\ &= 2^k - 2^{k2^{r-2}} + a \cdot 2^r + \sum_{i=2}^{r-1} 2^{k2^{r-i}} - \sum_{i=3}^r 2^{k2^{r-i}} \\ &= 2^k - 2^{k2^{r-2}} + a \cdot 2^r + 2^{k2^{r-2}} - 2^k + \sum_{i=3}^{r-1} 2^{k2^{r-i}} - \sum_{i=3}^{r-1} 2^{k2^{r-i}} \\ &= a \cdot 2^r \end{aligned}$$

and hence

$$a = 2^{k \cdot 2^{r-1} - r} \quad \text{if } r \geq 2.$$

This together with (6.27) implies for $r \geq 2$

$$\begin{aligned} [\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k)] &= 2^k \cdot [\mathbb{Q}] - 2^{k2^{r-2}-r+1} \cdot [\mathbb{Q}[G/G_1]] \\ &\quad + \sum_{i=2}^{r-1} (2^{k2^{r-i}-r+i} - 2^{k2^{r-1-i}-r+i}) \cdot [\mathbb{Q}[G/G_i]] + 2^{k \cdot 2^{r-1} - r} \cdot [\mathbb{Q}G]. \end{aligned}$$

This finishes the proof of assertion (ii).

(iii) One shows analogously to (6.25) for some integer b'

$$\begin{aligned} [\Lambda^{\text{alt}} \mathbb{Q}[\zeta]] &= [\mathbb{Q} \text{map}(G/G_1, P)_{\text{ev}}] + [\mathbb{Q} \text{map}(G/G_1, P)_{\text{odd}}] + b' \cdot [\mathbb{Q}G]. \\ &= [\mathbb{Q} \text{map}(G/G_1, P)] + b' \cdot [\mathbb{Q}G]. \end{aligned}$$

Again by the exponential law, we get

$$[\Lambda^{\text{alt}}(\mathbb{Q}[\zeta]^k)] = \prod_{l=1}^k [\Lambda^{\text{alt}} \mathbb{Q}[\zeta]].$$

Hence we get for some appropriate integer $b \in \mathbb{Z}$ in $R_{\mathbb{Q}}(G)$.

$$[\Lambda^{\text{alt}}(\mathbb{Q}[\zeta]^k)] = [\mathbb{Q} \text{map}(G/G_1, P)^k] + b \cdot [\mathbb{Q}G].$$

This implies by Lemma 6.3 (ii) applied to $H = G/G_1$

$$\begin{aligned} [\Lambda^{\text{alt}}(\mathbb{Q}[\zeta]^k)] &= p^k \cdot [\mathbb{Q}] + \sum_{i=0}^{r-2} (p^{kp^{r-i-1}-r+i+1} - p^{kp^{r-i-2}-r+i+1}) \cdot [\mathbb{Q}[G/G_{i+1}]] + b \cdot [\mathbb{Q}G] \\ &= p^k \cdot [\mathbb{Q}] + \sum_{i=1}^{r-1} (p^{kp^{r-i}-r+i} - p^{kp^{r-i-1}-r+i}) \cdot [\mathbb{Q}[G/G_i]] + b \cdot [\mathbb{Q}G]. \end{aligned}$$

Since

$$\sum_{j \geq 0} (-1)^j \cdot \dim_{\mathbb{Q}} (\Lambda^j(\mathbb{Q}[\zeta]^k)) = 0,$$

we get

$$b = -p^{k-r} - \sum_{i=1}^{r-1} (p^{kp^{r-i}-r} - p^{kp^{r-i-1}-r}) = -p^{kp^{r-1}-r}.$$

This finishes the proof of Lemma 6.24. \square

Lemma 6.28. *Suppose that $m = p^r$ for a prime p and an integer $r \geq 1$. Let k be a natural number uniquely determined by $n = k \cdot (p-1) \cdot p^{r-1}$. Then*

$$\sum_{l \geq 0} \text{rk}_{\mathbb{Z}}(\Lambda^l L)^G = \begin{cases} 1 + \frac{2^n - 1}{p^r} & \text{if } p \neq 2; \\ 2^{k-1} & \text{if } p = 2, r = 1; \\ 2^{2k-2} + 2^{k-1} & \text{if } p = 2, r = 2; \\ 2^{k-1} + 2^{k2^{r-2}-r+1} + 2^{k \cdot 2^{r-1}-r} + \sum_{i=3}^{r-1} 2^{k2^{r-i}-r+i-1} & \text{if } p = 2, r \geq 3. \end{cases}$$

Proof. Notice that $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic as $\mathbb{Q}G$ -module to $\mathbb{Q}[\zeta]^k$ for $\zeta = \exp(2\pi i/m)$. This implies

$$\begin{aligned} n &= k \cdot (p-1) \cdot p^{r-1}; \\ \sum_{l \geq 0} \text{rk}_{\mathbb{Z}}(\Lambda^l L)^G &= \dim_{\mathbb{Q}} \left((\Lambda^{\text{all}}(\mathbb{Q}[\zeta]^k))^G \right). \end{aligned}$$

Now the case $p \neq 2$ follows from Lemma 6.24 (i). Next we treat the case $p = 2$. Then the claim follows for $r = 1$ directly and for $r = 2$ and for $r \geq 3$ by the following calculations from Lemma 6.24 (ii). Namely, if $r = 2$, we get

$$\begin{aligned} \sum_{l \geq 0} \text{rk}_{\mathbb{Z}}(\Lambda^l L)^G &= 2^k - 2^{k2^{2-2}-2+1} + 2^{k \cdot 2^{2-1}-2} \\ &= 2^k - 2^{k-1} + 2^{2k-2} \\ &= 2^{2k-2} + 2^{k-1}. \end{aligned}$$

Suppose $r \geq 3$. Then

$$\begin{aligned}
 \sum_{l \geq 0} \text{rk}_{\mathbb{Z}}(\Lambda^l L)^G &= 2^k - 2^{k2^{r-2}-r+1} + 2^{k \cdot 2^{r-1}-r} + \sum_{i=2}^{r-1} (2^{k2^{r-i}-r+i} - 2^{k2^{r-1-i}-r+i}) \\
 &= 2^k - 2^{k2^{r-2}-r+1} + 2^{k \cdot 2^{r-1}-r} + \sum_{i=2}^{r-1} 2^{k2^{r-i}-r+i} - \sum_{i=2}^{r-1} 2^{k2^{r-1-i}-r+i} \\
 &= 2^k - 2^{k2^{r-2}-r+1} + 2^{k \cdot 2^{r-1}-r} + \sum_{i=2}^{r-1} 2^{k2^{r-i}-r+i} - \sum_{i=3}^r 2^{k2^{r-i}-r+i-1} \\
 &= 2^k - 2^{k2^{r-2}-r+1} + 2^{k \cdot 2^{r-1}-r} + 2^{k2^{r-2}-r+2} - 2^{k-1} \\
 &\quad + \sum_{i=3}^{r-1} 2^{k2^{r-i}-r+i} - \sum_{i=3}^{r-1} 2^{k2^{r-i}-r+i-1} \\
 &= 2^{k-1} + 2^{k2^{r-2}-r+1} + 2^{k \cdot 2^{r-1}-r} + \sum_{i=3}^{r-1} 2^{k2^{r-i}-r+i-1}.
 \end{aligned}$$

□

7. THE GENERAL CASE

In this section we consider the general case, where m is not necessarily a prime power.

Notation 7.1. We write

$$m = |G| = p_1^{r_1} \cdot p_2^{r_2} \cdots p_s^{r_s},$$

for distinct primes p_1, p_2, \dots, p_s and integers s, r_1, r_2, \dots, r_s , all of which are greater or equal to 1. If $C \subseteq G$ is a subgroup, denote by $C[i]$ the p_i -Sylow subgroup.

Given a subgroup $H \subseteq G[j]$, we denote in the sequel by H' the subgroup of $G = G[1] \times \cdots \times G[j-1] \times G[j] \times G[j+1] \times \cdots \times G[s]$ given by

$$H' := \{1\} \times \cdots \times \{1\} \times H \times \{1\} \cdots \times \{1\},$$

and analogously we define $H' \subseteq C$ for a subgroup H of $C[i]$. (Obviously $H \cong H'$.)

Lemma 7.2. *Let $C \subseteq G$ and $H \subseteq G$ be subgroups. Then we have*

$$H^1(C; L)^H \cong \{0\}$$

unless there exists $i \in \{1, 2, \dots, s\}$ such that $C[j] = H[j] = \{0\}$ holds for all $j \in \{1, 2, \dots, s\}$ with $j \neq i$.

Proof. Suppose that $H^1(C; L)^H \neq \{0\}$. Since $H^1(C; L)$ is annihilated by multiplication with m (see [4, Corollary 10.2 in III.10 on page 84]), there exists $i \in \{1, 2, \dots, s\}$ with

$$(H^1(C; L)^H)_{(p_i)} \neq 0.$$

We conclude for all $j \in \{1, 2, \dots, s\}$ because of $H[j]' \subseteq H$

$$(H^1(C; L)^{H[j]'})_{(p_i)} \neq 0.$$

Assume in the sequel $j \neq i$. Then p_i and $|H[j]'|$ are prime to one another and we conclude using [4, Theorem 10.3 in III.10 on page 84]

$$\begin{aligned}
 (H^1(C; L)^{H[j]'})_{(p_i)} &\cong (H^1(C; L)_{(p_i)})^{H[j]'} \\
 &\cong (H^1(C[i]; L)^{C/C[i]})^{H[j]'} .
 \end{aligned}$$

In particular

$$\left(H^1(C[i]; L)^{C/C[i]}\right)^{H[j]'} \neq \{0\}.$$

This implies $H^1(C[i]; L)^{C[j]'} \neq 0$ and $H^1(C[i]; L)^{H[j]'} \neq 0$. Since $|C[j]'|$ and $|C[i]|$ are prime to one another, we get

$$H^1(C[i]; L)^{C[j]'} \cong H^1(C[i]; L^{C[j]'}).$$

This implies $H^1(C[i]; L^{C[j]'}) \neq \{0\}$ and hence $L^{C[j]'} \neq \{0\}$. Since the G -action on L is free outside the origin, we conclude $C[j] = 0$.

Since $|H[j]|$ and $|C[i]|$ are prime to one another, we have

$$H^1(C[i]; L)^{H[j]'} \cong H^1(C[i]; L^{H[j]'}).$$

This implies $H^1(C[i]; L^{H[j]'}) \neq \{0\}$ and hence $L^{H[j]'} \neq 0$. Since the G -action on L is free outside the origin, we conclude $H[j] = 0$. \square

Let $C \subseteq G$ be a subgroup. Let

$$[H^1(C[i]; L)] \in A(G[i])$$

be the class of the finite $G[i]$ -set $H^1(C[i]; L)$, and let

$$[H^1(C; L)] \in A(G)$$

be the class of the finite G -set $H^1(C; L)$, where the actions come from the G -action on L . Denote by

$$\text{ind}_{G[i]}^G : A(G[i]) \rightarrow A(G)$$

the homomorphism of abelian groups coming from induction with the inclusion of groups $G[i] \rightarrow G$.

Lemma 7.3. *Suppose that there exists $i \in \{1, 2, \dots, s\}$ such that $C[j] = \{0\}$ holds for all $j \in \{1, 2, \dots, s\}$ with $j \neq i$. Then we get in $A(G)$*

$$\text{ind}_{G[i]}^G ([H^1(C[i]; L)] - [G[i]/G[i]]) = \frac{m}{p_i^{r_i}} \cdot ([H^1(C; L)] - [G/G]).$$

Proof. We have the isomorphism of G -sets

$$\begin{aligned} \text{ind}_{G[i]}^G (H^1(C[i]; L)) \\ \cong G[1] \times \cdots \times G[i-1] \times H^1(C[i]; L) \times G[i+1] \times \cdots \times G[r]. \end{aligned}$$

Consider a subgroup $H \subseteq G$. If $H[j] = \{0\}$ for each $j \in \{1, \dots, s\}$ with $j \neq i$, then we get for every finite $G[i]$ -set S

$$\begin{aligned} \text{ch}_H^G (\text{ind}_{G[i]}^G ([S])) &= |(\text{ind}_{G[i]}^G S)^H| \\ &= |G[1] \times \cdots \times G[i-1] \times S^{H[i]} \times G[i+1] \times \cdots \times G[r]| \\ &= |G[1]| \cdots |G[i-1]| \cdot |S^{H[i]}| \cdot |G[i+1]| \cdots |G[r]| \\ &= \frac{m}{p_i^{r_i}} \cdot |S^{H[i]}|, \end{aligned}$$

and hence because of $C = C[i]$ and $H = H[i]$

$$\begin{aligned}
 (7.4) \quad \text{ch}_H^G \left(\text{ind}_{G[i]}^G ([H^1(C[i]; L)] - [G[i]/G[i]]) \right) \\
 &= \frac{m}{p_i^{r_i}} \cdot \left(|H^1(C[i]; L)^{H[i]}| - |G[i]/G[i]^{H[i]}| \right) \\
 &= \frac{m}{p_i^{r_i}} \cdot \left(|H^1(C[i]; L)^{H[i]}| - 1 \right) \\
 &= \frac{m}{p_i^{r_i}} \cdot (|H^1(C; L)^H| - |G/G^H|) \\
 &= \frac{m}{p_i^{r_i}} \cdot \text{ch}_H^G ([H^1(C; L)] - [G/G]).
 \end{aligned}$$

Suppose that there exists $j \in \{1, \dots, s\}$ with $j \neq i$ and $H[j] \neq \{0\}$. Then we get for every finite $G[i]$ -set S

$$\begin{aligned}
 &\text{ch}_H^G (\text{ind}_{G[i]}^G ([S])) \\
 &= |(\text{ind}_{G[i]}^G S)^H| \\
 &= |G[1]^{H[1]} \times \dots \times G[i-1]^{H[i-1]} \times S^{H[i]} \times G[i+1]^{H[i+1]} \times \dots \times G[r]^{H[r]}| \\
 &= |\emptyset| \\
 &= 0,
 \end{aligned}$$

and we compute using Lemma 7.2

$$\begin{aligned}
 (7.5) \quad \text{ch}_H^G \left(\text{ind}_{G[i]}^G ([H^1(C[i]; L)] - [G[i]/G[i]]) \right) \\
 &= 0 \\
 &= \frac{m}{p_i^{r_i}} \cdot (|H^1(C; L)^H| - 1) \\
 &= \frac{m}{p_i^{r_i}} \cdot \text{ch}_H^G ([H^1(C; L)] - [G/G]).
 \end{aligned}$$

Since the character map

$$\text{ch}^G: A(G) \rightarrow \prod_{H \subseteq G} \mathbb{Z}, \quad [S] \mapsto |S^H|$$

is injective, Lemma 7.3 follows from (7.4) and (7.5). \square

Theorem 7.6 (General case). *Then we get using Notation 7.1:*

(i) *There exists an natural number k uniquely determined by the property*

$$n = k \cdot \prod_{i=1}^s (p_i - 1) \cdot p_i^{r_i - 1};$$

(ii) *Suppose that there exists $i \in \{1, 2, \dots, s\}$ such that $C[j] = \{0\}$ holds for all $j \in \{1, 2, \dots, s\}$ with $j \neq i$. Define the natural numbers r_i and c_i by $|G[i]| = p_i^{r_i}$ and $|C[i]| = p_i^{c_i}$. Let $k[i]$ be the integer uniquely determined by $n = (p_i - 1) \cdot p_i^{r_i - 1} \cdot k[i]$. (Its existence follows from assertion (i).) Let $G[i]_l$ be the subgroup of $G[i]$ of order p_i^l .*

Then we get in $A(G)$

$$\begin{aligned}
 [H^1(C; L)] &= [G/G] + \frac{(p_i^{k[i]} - 1) \cdot p_i^{r_i}}{m} \cdot [G/G[i]'] \\
 &+ \sum_{l=0}^{r_i - c_i - 1} \frac{p_i^{k[i]p_i^{r_i - c_i - l} + c_i + l} - p_i^{k[i]p_i^{r_i - c_i - l - 1} + c_i + l}}{m} \cdot [G/G[i]']_{c_i + l};
 \end{aligned}$$

(iii) Suppose that there exists no $i \in \{1, 2, \dots, s\}$ such that $C[j] = \{0\}$ holds for all $j \in \{1, 2, \dots, s\}$ with $j \neq i$. Then we get in $A(G)$.

$$[H^1(C; L)] = [G/G].$$

(iv)

(a) If $C = G$, we have

$$|\mathcal{M}(C)| = |\mathcal{C}(C)| = \begin{cases} p^k & s = 1; \\ 1 & s \geq 2; \end{cases}$$

(b) If C is different from G and is not a p -group, then

$$\begin{aligned} |\mathcal{C}(C)| &= 1; \\ |\mathcal{M}(C)| &= 0; \end{aligned}$$

(c) If C is different from G and $|C| = p_i^j$ for some $i \in \{1, 2, \dots, s\}$ and $j \in \{1, 2, \dots, r_i\}$, then

$$|\mathcal{M}(C)| = \begin{cases} \frac{(p_i^{k[i]} - 1) \cdot p_i^{r_i}}{m} & j = r_i; \\ \frac{p_i^{k[i]p_i^{r_i-j} + j} - p_i^{k[i]p_i^{r_i-j-1} + j}}{m} & j \leq r_i - 1. \end{cases}$$

Proof. (i) The rational G -representation $L \otimes_{\mathbb{Z}} \mathbb{Q}$ has by assumption the property that $(L \otimes_{\mathbb{Z}} \mathbb{Q})^C = \{0\}$ for all subgroups $C \subseteq G$ with $C \neq \{0\}$. Hence there exists a natural number k such that $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is $\mathbb{Q}G$ -isomorphic to $\mathbb{Q}[\exp(2\pi i/m)]^k$ (see [22, Exercise 13.1 on page 104]). Since $n = \dim_{\mathbb{Q}}(L \otimes_{\mathbb{Z}} \mathbb{Q})$ and $\dim_{\mathbb{Q}}(\mathbb{Q}[\exp(2\pi i/m)]) = \prod_{i=1}^s (p_i - 1) \cdot p_i^{r_i-1}$, assertion (i) follows.

(ii) From Lemma 7.3 we obtain the following equality in $A(G)$

$$[H^1(C; L)] = [G/G] - \frac{p_i^{r_i}}{m} \cdot \text{ind}_{G[i]}^G ([G[i]/G[i]]) + \frac{p_i^{r_i}}{m} \cdot \text{ind}_{G[i]}^G ([H^1(C[i]; L)]).$$

We get from Theorem 6.18 applied to $G[i]$ the following equation in $A(G[i])$

$$[H^1(C[i]; L)] = p^{k[i]} \cdot [G[i]/G[i]] + \sum_{l=0}^{r_i - c_i - 1} \frac{p_i^{k[i]p_i^{r_i - c_i - l}} - p_i^{k[i]p_i^{r_i - c_i - 1 - l}}}{p_i^{r_i - c_i - l}} \cdot [G[i]/G_{c_j+l}].$$

Since $\text{ind}_{G[i]}^G ([G[i]/G_{c_j+l}]) = G/G[i]_{c_i+l}'$, assertion (ii) follows.

(iii) By Lemma 7.2 we have $H^1(C; L) = \{0\}$ and hence $[H^1(C; L)] = [G/G]$.

(iv) Claim (iv)a follows from Theorem 6.18 if G is a p -group. If G is not a p -group, claim (iv)a follows from Theorem 5.8 (iv) since $|H^1(G; L)| = 1$ holds by assertion (iii). The claim (iv)b from Theorem 5.8 (i) and (iii) since $|H^1(C; L)| = 1$ holds by assertion (iii).

It remains to prove claim (iv)c. We begin with the case $j = r_i$. Then we get from Theorem 5.8 (iii) and assertions (ii), (iv)a and (iv)b

$$\begin{aligned} |\mathcal{C}(C)| &= 1 + \frac{(p_i^{k[i]} - 1) \cdot p_i^{r_i}}{m} \\ |\mathcal{C}(D)| &= 1 \end{aligned} \quad \text{for } C \subsetneq D \subseteq G.$$

Now apply Theorem 5.8 (i). Finally we consider the case $j \leq r_i - 1$. Let D be an subgroup of G with $C \subsetneq D$. If D is not a p -group, we get $|\mathcal{C}(D)| = 1$ from Theorem 5.8 (iii) and assertion (iii). Hence the image of $\mathcal{C}(D) \rightarrow \mathcal{C}(C)$ is contained in the image of $\mathcal{C}(G) \rightarrow \mathcal{C}(C)$. Let $C_l \subseteq G$ be the subgroup of order p_i^l for $l = 0, 1, 2, \dots, r_i$. Then for every subgroup $D \subseteq G$ with $C \subsetneq D$ the image of

$\mathcal{C}(D) \rightarrow \mathcal{C}(C)$ is contained in the image of $\mathcal{C}(C_{j+1}) \rightarrow \mathcal{C}(C)$. We conclude from Theorem 5.8 (i) and (iii) and assertion (ii)

$$\begin{aligned}
 |\mathcal{M}(C)| &= |\mathcal{C}(C)| - |\mathcal{C}(C_{j+1})| \\
 &= 1 + \frac{(p_i^{k[i]} - 1) \cdot p_i^{r_i}}{m} + \sum_{l=0}^{r_i-j-1} \frac{p_i^{k[i]p_i^{r_i-j-l}+j+l} - p_i^{k[i]p_i^{r_i-j-l-1}+j+l}}{m} \\
 &\quad - 1 - \frac{(p_i^{k[i]} - 1) \cdot p_i^{r_i}}{m} - \sum_{l=0}^{r_i-j-2} \frac{(p_i^{k[i]p_i^{r_i-j-1-l}+j+1+l} - p_i^{k[i]p_i^{r_i-j-1-l-1}+j+1+l})}{m} \\
 &= \sum_{l=0}^{r_i-j-1} \frac{p_i^{k[i]p_i^{r_i-j-l}+j+l} - p_i^{k[i]p_i^{r_i-j-l-1}+j+l}}{m} \\
 &\quad - \sum_{l=1}^{r_i-j-1} \frac{p_i^{k[i]p_i^{r_i-j-l}+j+l} - p_i^{k[i]p_i^{r_i-j-l-1}+j+l}}{m} \\
 &= \frac{p_i^{k[i]p_i^{r_i-j}+j} - p_i^{k[i]p_i^{r_i-j-1}+j}}{m}.
 \end{aligned}$$

□

Example 7.7 (m square-free). Suppose that m is square-free, or, equivalently, $r_1 = r_2 = \dots = r_s = 1$. Let k be the natural number uniquely determined by $n = k \cdot \prod_{j=1}^s (p_j - 1)$. If $s = 1$, then every non-trivial finite subgroup of Γ is cyclic of order p and maximal among finite subgroups and we conclude from Theorem 6.18

$$|\mathcal{M}| = p^k.$$

If $s \geq 2$, then

$$|\mathcal{M}(C)| = \begin{cases} 1 & C = G; \\ 0 & C \neq G \text{ and } C \neq G[i] \text{ for every } i \in \{1, 2, \dots, s\}; \\ \frac{p_i \cdot (p_i^{n/(p_i-1)} - 1)}{m} & C = G[i] \text{ for some } i \in \{1, 2, \dots, s\}. \end{cases}$$

Remark 7.8. It follows from the proof that all fractions appearing Theorem 7.6 are indeed natural numbers. As an illustration we check this in Example 7.7. We have to show that for every $j \in \{1, 2, \dots, s\}$ with $j \neq i$ the prime number p_j divides $p_i^{k \cdot \prod_{j \in \{1, 2, \dots, s\}, j \neq i} (p_j - 1)} - 1$. Now the claim follows from Fermat's little theorem, i.e., from $p_i^{p_j-1} \equiv 1 \pmod{p_j}$.

Lemma 7.9. *Suppose that m is even. Then*

$$(\Lambda^{\text{odd}} \mathbb{Q}[\zeta_m])^G = 0.$$

Proof. Since G is even, it contains an element g of order 2. This element acts by $-\text{id}$ in $\mathbb{Q}[\zeta_m]$. Hence it acts by $-\text{id}$ on $\Lambda^{2l+1} \mathbb{Q}[\zeta_m]$ for every $l \geq 0$. This implies $\Lambda^{2l+1} \mathbb{Q}[\zeta_m]^{(g)} = \{0\}$ for every $l \geq 0$ and hence $(\Lambda^{\text{odd}} \mathbb{Q}[\zeta_m])^G = 0$. □

Remark 7.10 (Computing s_i for even m). Lemma 7.9 implies for even m that

$$\begin{aligned}
 \sum_{l \in \mathbb{Z}} \text{rk}_{\mathbb{Z}}((\Lambda^{2l+1} \mathbb{Z}^n)^{\mathbb{Z}/m}) &= 0; \\
 \sum_{l \in \mathbb{Z}} \text{rk}_{\mathbb{Z}}((\Lambda^{2l} \mathbb{Z}^n)^{\mathbb{Z}/m}) &= \sum_{l \in \mathbb{Z}} (-1)^l \cdot \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}^n)^{\mathbb{Z}/m}).
 \end{aligned}$$

The alternating sum $\sum_{l \in \mathbb{Z}} (-1)^l \cdot \text{rk}_{\mathbb{Z}}((\Lambda^l \mathbb{Z}^n)^{\mathbb{Z}/m})$ will be computed in terms of \mathcal{M} in Theorem 8.7 (ii) (see Remark 8.9). Hence we will be able to compute the numbers

s_i appearing in Theorem 0.1 (iii) explicitly using Theorem 7.6 (iv) provided that m is even.

8. EQUIVARIANT EULER CHARACTERISTICS

Recall that we are considering the extension $1 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{\pi} \mathbb{Z}/m \rightarrow 1$ as it appears in Theorem 0.1 and we will abbreviate $L = \mathbb{Z}^n$ and $G = \mathbb{Z}/m$.

Given a finite G -CW-complex, define its *G-Euler characteristic*

$$(8.1) \quad \chi^G(X) := \sum_c (-1)^{d(c)} \cdot [G/G_c] \in A(G)$$

where c runs over the equivariant cells $G/G_c \times D^{d(c)}$.

Theorem 8.2 (Equivariant Euler characteristic of $L \setminus \underline{E}\Gamma$).
We get in the Burnside ring $A(G)$

$$\chi^G(L \setminus \underline{E}\Gamma) = a \cdot [G] + \sum_{(M) \in \mathcal{M}} [G/\pi(M)],$$

where the integer a is given by $-\sum_{(M) \in \mathcal{M}} \frac{1}{|M|}$.

Proof. Since the G -action on L is free outside the origin, we obtain from Lemma 5.1 and [20, Corollary 2.11] a cellular Γ -pushout

$$\begin{array}{ccc} \coprod_{(M) \in \mathcal{M}} \Gamma \times_M EM & \xrightarrow{i_0} & E\Gamma \\ \downarrow \coprod_{(M) \in \mathcal{M}} \text{pr}_M & & \downarrow f \\ \coprod_{(M) \in \mathcal{M}} \Gamma/M & \xrightarrow{i_1} & X \end{array}$$

such that X is a model for $\underline{E}\Gamma$. There exists a finite Γ -CW-model Y for $\underline{E}\Gamma$ by Lemma 3.1. Choose a Γ -homotopy equivalence $f: Y \rightarrow X$. Let Z be defined by the Γ -pushout

$$\begin{array}{ccc} Y^{>\{1\}} & \longrightarrow & Y \\ \downarrow f^{>\{1\}} & & \downarrow \\ X^{>\{1\}} & \longrightarrow & Z \end{array}$$

Then Z is a finite Γ -CW-model for $\underline{E}\Gamma$ with $Z^{>\{1\}} = X^{>\{1\}} = \coprod_{(M) \in \mathcal{M}} \Gamma/M$. In the sequel we write $\underline{E}\Gamma$ for Z . Then $L \setminus \underline{E}\Gamma$ is a finite G -CW-complex with

$$(L \setminus \underline{E}\Gamma)^{>\{1\}} = \coprod_{(M) \in \mathcal{M}} G/\pi(M).$$

Hence we get for some integer a in $A(G)$

$$\chi^G(L \setminus \underline{E}\Gamma) = a \cdot [G] + \sum_{(M) \in \mathcal{M}} [G/\pi(M)].$$

Let $\text{ch}_{\{1\}}^G: A(G) \rightarrow \mathbb{Z}$ be the map sending the class of a finite G -set to its cardinality. It sends $\chi^G(L \setminus \underline{E}\Gamma)$ to the (non-equivariant) Euler characteristic of $L \setminus \underline{E}\Gamma$ which is

zero since $L \setminus \underline{E}\Gamma$ is homotopy equivalent to the n -torus. Hence we get

$$\begin{aligned} 0 &= \text{ch}_{\{1\}}^G(\chi^G(L \setminus \underline{E}\Gamma)) \\ &= \text{ch}_{\{1\}}^G\left(a \cdot [G] + \sum_{(M) \in \mathcal{M}} [G/\pi(M)]\right) \\ &= a \cdot |G| + \sum_{(M) \in \mathcal{M}} |G/\pi(M)| \\ &= |G| \cdot \left(a + \sum_{m \in \mathcal{M}} \frac{1}{|M|}\right). \end{aligned}$$

This implies $a = -\sum_{(M) \in \mathcal{M}} \frac{1}{|M|}$. \square

Theorem 8.3 (Rational permutation modules).

(i) Sending a finite G -set to the associated $\mathbb{Q}G$ -permutation module defines an isomorphism of rings

$$\text{perm}: A(G) \rightarrow R_{\mathbb{Q}}(G);$$

(ii) If X is a finite G -CW-complex, then

$$\text{perm}(\chi^G(X)) = \sum_{i \geq 0} (-1)^i \cdot [H_i(X; \mathbb{Q})] = [K_0(X) \otimes_{\mathbb{Z}} \mathbb{Q}] - [K_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}].$$

We have

$$\text{perm}(\chi^G(L \setminus \underline{E}\Gamma)) = \sum_{i \geq 0} (-1)^i \cdot [\Lambda^i L \otimes_{\mathbb{Z}} \mathbb{Q}].$$

Proof. (i) The map perm is surjective by [22, Exercise 13.1 (c) on page 105]. Since the source and target of perm are finitely generated free abelian groups of the same rank, perm is bijective.

(ii) We compute in $R_{\mathbb{Q}}(G)$, where $C_*(X)$ is the cellular \mathbb{Z} -chain complex of X

$$\begin{aligned} \text{perm}(\chi^G(X)) &= \sum_{i \geq 0} (-1)^i \cdot [C_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}] \\ &= \sum_{i \geq 0} (-1)^i \cdot [H_i(C_*(X) \otimes_{\mathbb{Z}} \mathbb{Q})] = \sum_{i \geq 0} (-1)^i \cdot [H_i(X; \mathbb{Q})]. \end{aligned}$$

The homological Chern character [8] yields an isomorphism of $\mathbb{Q}G$ -modules

$$(8.4) \quad \bigoplus_{j \in \mathbb{Z}} H_{i+2j}(X; \mathbb{Q}) \xrightarrow{\cong} K_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

The cup product induces natural isomorphisms

$$\Lambda_{\mathbb{Q}}^i H^1(L \setminus \underline{E}\Gamma, \mathbb{Q}) \xrightarrow{\cong} H^i(L \setminus \underline{E}\Gamma, \mathbb{Q}),$$

since $L \setminus \underline{E}\Gamma$ is homotopy equivalence to the n -torus. There are natural isomorphisms $L \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H_1(L \setminus \underline{E}\Gamma; \mathbb{Q})$, $\Lambda_{\mathbb{Q}}^i((L \otimes_{\mathbb{Z}} \mathbb{Q})^*) \xrightarrow{\cong} (\Lambda^i L \otimes_{\mathbb{Z}} \mathbb{Q})^*$ and $H_i(L \setminus \underline{E}\Gamma; \mathbb{Q})^* \xrightarrow{\cong} H^i(L \setminus \underline{E}\Gamma; \mathbb{Q})$. For every finitely generated $\mathbb{Q}G$ -module M there is a canonical $\mathbb{Q}G$ -isomorphism $M \xrightarrow{\cong} (M^*)^*$. This implies that the $\mathbb{Q}G$ -modules $H_i(L \setminus \underline{E}\Gamma; \mathbb{Q})$ and $\Lambda^i L \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic. \square

Lemma 8.5. Let X be a finite G -CW-complex. The map of abelian groups

$$\text{quot}: A(G) \rightarrow \mathbb{Z}, \quad [S] \mapsto |G \setminus S|$$

sends the G -Euler characteristic $\chi^G(X)$ to the Euler characteristic $\chi(G \backslash X)$ of the quotient. We have

$$\chi(G \backslash X) = \dim_{\mathbb{Q}}(K_0(G \backslash X) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1(G \backslash X) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Proof. We get $\text{quot}(\chi^G(X)) = \chi(G \backslash X)$ by inspecting the definitions of the Euler characteristics in terms of counting (equivariant) cells. The second claim follows from the homological Chern character (8.4). \square

Lemma 8.6. *Let X be a finite G -CW-complex. The map of abelian groups*

$$\text{kg}: A(G) \rightarrow \mathbb{Z}, \quad [G/H] \mapsto |H|$$

satisfies

$$\text{kg}(\chi^G(X)) = \dim_{\mathbb{Q}}(K_0^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Proof. The expression

$$\chi_{K_*^G}(X) := \dim_{\mathbb{Q}}(K_0^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1^G(X) \otimes_{\mathbb{Z}} \mathbb{Q})$$

depends only on the G -homotopy type of X and is additive under G -pushouts of finite G -CW-complexes. The latter follows from the Mayer-Vietoris sequence associated to such a G -pushout and Bott periodicity. Hence $\chi_{K_*^G}(X)$ can be computed by counting equivariant cells, where the contribution of an equivariant cell of the type $G/H \times D^d$ is

$$\begin{aligned} \chi_{K_*^G}(G/H \times D^d) &= (-1)^d \cdot \chi_{K_*^G}(G/H) \\ &= (-1)^d \cdot (\dim_{\mathbb{Q}}(K_0^G(G/H) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1^G(G/H) \otimes_{\mathbb{Z}} \mathbb{Q})) \\ &= (-1)^d \cdot (\dim_{\mathbb{Q}}(K_0^H(\text{pt}) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1^H(\text{pt}) \otimes_{\mathbb{Z}} \mathbb{Q})) \\ &= (-1)^d \cdot (\dim_{\mathbb{Q}}(R_{\mathbb{C}}(H) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(\{0\} \otimes_{\mathbb{Z}} \mathbb{Q})) \\ &= (-1)^d \cdot |H|. \end{aligned}$$

\square

Theorem 8.7 (A priori estimates).

(i) *We obtain the following identity in $R_{\mathbb{Q}}(G)$*

$$\left(- \sum_{(M) \in \mathcal{M}} \frac{1}{|M|} \right) \cdot [\mathbb{Q}G] + \sum_{(M) \in \mathcal{M}} \mathbb{Q}[G/\pi(M)] = \sum_{l \geq 0} (-1)^l \cdot [\Lambda^l L \otimes_{\mathbb{Z}} \mathbb{Q}];$$

(ii) *We have the following identity of integers*

$$\begin{aligned} \sum_{(M) \in \mathcal{M}} \frac{|M| - 1}{|M|} = \chi(\underline{B}\Gamma) &= \dim_{\mathbb{Q}}(K_0(\underline{B}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1(\underline{B}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &= \sum_{l \in \mathbb{Z}} (-1)^l \cdot \text{rk}_{\mathbb{Z}}((\Lambda^l L)^G); \end{aligned}$$

(iii) *We have the following identity of integers*

$$\begin{aligned} \sum_{(M) \in \mathcal{M}} \frac{|M|^2 - 1}{|M|} &= \dim_{\mathbb{Q}}(K_0^G(L \backslash \underline{E}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1^G(L \backslash \underline{E}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &= \dim_{\mathbb{Q}}(K_0(C_r^*(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1(C_r^*(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}). \end{aligned}$$

Proof. (i) This follows from Theorem 8.2 and Theorem 8.3

(ii) This follows from Theorem 8.2, Theorem 8.3 and Lemma 8.5 as soon as we have shown

$$(8.8) \quad \dim_{\mathbb{Q}}(K_0(\underline{B}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim_{\mathbb{Q}}(K_1(\underline{B}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}) = \sum_{l \in \mathbb{Z}} (-1)^l \cdot \text{rk}_{\mathbb{Z}}((\Lambda^l L)^G).$$

Since G acts properly on $L \backslash \underline{E}\Gamma$, there is an isomorphism of \mathbb{Q} -modules

$$K_i(\underline{B}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (K_i(L \backslash \underline{E}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q})^G$$

Now (8.8) follows from Theorem 8.3 (ii).

(iii) The first equation follows from Theorem 8.2 and Lemma 8.6. The second equation is a direct consequence of the Baum-Connes Conjecture, which is known to be true for Γ (see [10]), and from the isomorphism $K_i^{\Gamma}(\underline{E}\Gamma) \xrightarrow{\cong} K_i^G(L \backslash \underline{E}\Gamma)$ coming from the fact that L acts freely on $\underline{E}\Gamma$. \square

Remark 8.9. Recall that we can write \mathcal{M} as the disjoint union

$$\mathcal{M} = \coprod_{\{1\} \subsetneq C \subseteq G} \mathcal{M}(C).$$

This implies

$$\begin{aligned} \sum_{(M) \in \mathcal{M}} \frac{1}{|M|} &= \sum_{\{1\} \subsetneq C \subseteq G} \frac{|\mathcal{M}(C)|}{|C|}; \\ \sum_{(M) \in \mathcal{M}} \mathbb{Q}[G/\pi(M)] &= \sum_{\{1\} \subsetneq C \subseteq G} |\mathcal{M}(C)| \cdot [\mathbb{Q}[G/C]]; \\ \sum_{(M) \in \mathcal{M}} \frac{|M| - 1}{|M|} &= \sum_{\{1\} \subsetneq C \subseteq G} |\mathcal{M}(C)| \cdot \frac{|C| - 1}{|C|}; \\ \sum_{(M) \in \mathcal{M}} \frac{|M|^2 - 1}{|M|} &= \sum_{\{1\} \subsetneq C \subseteq G} |\mathcal{M}(C)| \cdot \frac{|C|^2 - 1}{|C|}. \end{aligned}$$

Hence one can compute the sums of the right sides of the equations above if one knows all the numbers $|\mathcal{M}(C)|$. These have been computed explicitly in Theorem 7.6 (iv).

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